A Generalization of the Łoś-Tarski Preservation Theorem via Characterizations of Σ^0_n and Π^0_n Theories

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Preservation theorems in First Order logic (henceforth called FO) are amongst the classical areas of study in model theory. One of the earliest preservation theorems (from 1949-50) is the Łoś-Tarski theorem, which gives a semantic characterization of Σ^0_1 theories (resp. Π^0_1 theories) in terms of preservation under extensions (resp. substructures). Subsequently, various characterizations of Π^0_2 theories were discovered, in terms of preservation under unions of ascending chains, interesections of descending chains, intersections of submodels, etc. Eventually, a uniform set of preservation theorems characterizing Σ^0_n and Π^0_n theories for all $n \geq 1$ was provided by Keisler [3] in 1960, by introducing the notion of *sandwiches of finite orders.* To our knowledge, these theorems are the only characterizations in the literature, of the aforementioned theories, that further seamlessly generalize the Łoś-Tarski theorem.

In this paper, we provide a new family of preservation theorems that characterize Σ^0_n and Π^0_n theories for all $n \geq 1$, and that seamlessly generalize the Łoś-Tarski theorem (see Theorem 0.5). Our semantic properties are simpler and easier to state than those defined by Keisler in [3], and vis-à-vis the latter properties, the former can be directly seen to generalize the preservation properties of the Łoś-Tarski theorem. Further, our proofs are much simpler than those of Keisler, and make use of natural generalizations of the ideas contained in the standard proofs of the Łoś-Tarski theorem.

We assume that the reader is familiar with usual notation and terminology used in the syntax and semantics of FO [1]. For $n > 0$, we denote by Σ^0_n (resp. Π^0_n), all FO sentences in prenex normal form, whose quantifier prefix begins with a $\exists$ (resp. $\forall$) and consists of at most $n-1$ alternations of quantifiers. A theory is a set of sentences. A Σ^0_n (resp. Π^0_n) theory is a set of Σ^0_n (resp. Π^0_n) sentences. We will assume familiarity with the model-theoretic notions of isomorphisms, substructures, extensions, elementary substructures and elementary extensions. Given structures $\mathfrak{A}$ and $\mathfrak{B}$, we denote by $\mathfrak{A} \cong \mathfrak{B}$ that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$, by $\mathfrak{A} \subseteq \mathfrak{B}$ that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and by $\mathfrak{A} \preceq \mathfrak{B}$, that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$. We will denote by $\mathfrak{A} \Rrightarrow_n \mathfrak{B}$, that each Σ^0_n sentence that is true in $\mathfrak{A}$ is also true in $\mathfrak{B}$.

0.1 New Characterizations of Σ^0_n and Π^0_n Theories

A theory T is *preserved under extensions (resp. substructures)* if whenever $\mathfrak{A}$ models T and $\mathfrak{B}$ is an extension (resp. substructure) of $\mathfrak{A}$, then $\mathfrak{B}$ models T. Theorem 0.1 syntactically characterizes such theories T. The proof of Theorem 0.1 uses Theorem 0.2 in a crucial way.

Theorem 0.1 (Łoś-Tarski, 1949-50, ref. [2]) *A theory T is preserved under extensions (resp. substructures) iff T is equivalent to a Σ^0_1 theory (resp. Π^0_1 theory).*

Theorem 0.2 (Existential Amalgamation Theorem, ref. [2]) *Let $\mathfrak{A}, \mathfrak{B}$ be τ-structures. Then $\mathfrak{A} \Rrightarrow_1 \mathfrak{B}$ iff there exist structures $\mathfrak{C}$ and $\mathfrak{D}$ s.t. (i) $\mathfrak{B} \preceq \mathfrak{D}$ (ii) $\mathfrak{C} \cong \mathfrak{A}$ and (iii) $\mathfrak{C} \subseteq \mathfrak{D}$.*

We now recall the notions of Σ^0_n-*extensions* and Σ^0_n-*substructures* from the literature.

Definition 0.3 (ref. [1]) *Given a positive integer n and structures $\mathfrak{A}$ and $\mathfrak{B}$, we say $\mathfrak{B}$ is a* Σ^0_n-extension *of $\mathfrak{A}$, denoted $\mathfrak{A} \subseteq_n \mathfrak{B}$ if (i) $\mathfrak{A} \subseteq \mathfrak{B}$ and (ii) for every Σ^0_n formula $\varphi(x_1, \ldots, x_k)$ and every k-tuple $\bar{a}$ from $\mathfrak{A}$, if $\mathfrak{A} \models \varphi(\bar{a})$, then $\mathfrak{B} \models \varphi(\bar{a})$. If $\mathfrak{A} \subseteq_n \mathfrak{B}$, then we say $\mathfrak{A}$ is a* Σ^0_n-substructure *of $\mathfrak{B}$.*

Observe that $\mathfrak{A} \subseteq_1 \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$. Also, $\mathfrak{A} \preceq \mathfrak{B}$ iff $\mathfrak{A} \subseteq_n \mathfrak{B}$ for all $n \in \mathbb{N}$. We now present a generalization of Theorem 0.2, that we call as the Σ^0_n Amalgamation Theorem. Observe that the former is special case of the latter for $n = 1$.

Theorem 0.4 (Σ^0_n Amalgamation Theorem) *Let $\mathfrak{A}, \mathfrak{B}$ be τ-structures. For each $n \geq 1$, $\mathfrak{A} \Rrightarrow_n \mathfrak{B}$ iff there exist structures $\mathfrak{C}$ and $\mathfrak{D}$ s.t. (i) $\mathfrak{B} \preceq \mathfrak{D}$ (ii) $\mathfrak{C} \cong \mathfrak{A}$ and (iii) $\mathfrak{C} \subseteq_n \mathfrak{D}$.*

Proof Sketch: Let $\tau_{\mathfrak{A}}$ be the vocabulary of $\mathfrak{A}$ expanded with $|\mathfrak{A}|$ constants, one constant per element of $\mathfrak{A}$, and let $\mathfrak{A}_{\mathfrak{A}}$ be the natural expansion of $\mathfrak{A}$ into a $\tau_{\mathfrak{A}}$-structure. Let $S_{(\Pi,n-1)}(\mathfrak{A}_{\mathfrak{A}})$ be the Π_{n-1} theory of $\mathfrak{A}_{\mathfrak{A}}$ and *El-diag*$(\mathfrak{B})$ be the elementary diagram of $\mathfrak{B}$. Show that *El-diag*$(\mathfrak{B}) \cup S_{(\Pi,n-1)}(\mathfrak{A}_{\mathfrak{A}})$ is satisfiable. ■

Define *preservation under Σ^0_n-extensions (resp. Σ^0_n-substructures)* similar to preservation under extensions (resp. substructures) as presented above. The central result of the paper can now be stated as below. Observe that the case of $n = 1$ in Theorem 0.5 is exactly Theorem 0.1.

Theorem 0.5 *For each $n \geq 1$, a theory T is preserved under Σ^0_n-extensions (resp. Σ^0_n-substructures) iff T is equivalent to a Σ^0_n theory (resp. Π^0_n theory).*

Proof Sketch: We sketch the proof for theories T preserved under Σ^0_n-extensions. Let Γ be the set of all Σ^0_n consequences of T. Clearly T entails Γ. Towards the converse, let $\mathfrak{B}$ model Γ. Show that the theory $T \cup S_{(\Pi,n)}(\mathfrak{B})$ is satisfied by a structure say $\mathfrak{A}$. Then $\mathfrak{A} \Rrightarrow_n \mathfrak{B}$. Now use Theorem 0.4 and the fact that T is preserved under Σ^0_n-extensions. ■

0.2 Comparison with Keisler's characterizations

For $0 < n < \omega$, a sequence of structures $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is said to be a *sandwich of order n* if (i) $\mathfrak{A}_i \subseteq \mathfrak{A}_{i+1}$ for $i + 1 \leq n$ and (ii) $\mathfrak{A}_i \preceq \mathfrak{A}_{i+2}$ for $i + 2 \leq n$. Given a structure $\mathfrak{A}$, define the classes n-Sand$(\mathfrak{A})$ and n-Sand-by$(\mathfrak{A})$, of structures as follows: (i) $\mathfrak{B} \in n$-Sand$(\mathfrak{A})$ iff $\mathfrak{B}$ and $\mathfrak{A}$ are (in order) the first two elements of a sandwich of order n (ii) $\mathfrak{B} \in n$-Sand-by$(\mathfrak{A})$ iff for some elementary extension $\mathfrak{B}'$ of $\mathfrak{B}$, it is the case that $\mathfrak{A}$ and $\mathfrak{B}'$ are (in order) the first two elements of a sandwich of order n. A theory T is *preserved under n-Sand (resp. n-Sand-by)* if for all models $\mathfrak{A}$ of T, if $\mathfrak{B}$ belongs to n-Sand$(\mathfrak{A})$ (resp. n-Sand-by$(\mathfrak{A})$), then $\mathfrak{B}$ models T.

Theorem 0.6 (Keisler, 1960, ref. [3]) *For each $n \geq 1$, a theory T is preserved under n-Sand-by (resp. n-Sand) iff T is equivalent to a Σ^0_n theory (resp. Π^0_n theory).*

It is clear that the preservation properties mentioned in Theorem 0.5 are simpler and easier to state than those in Theorem 0.6 above. As for the proofs, a reader who is familiar with the (fairly easy) standard proofs of Theorems 0.2 and 0.1 will identify that the proofs of Theorems 0.4 and 0.5 proceed along very similar lines as the former respectively. The proof of Theorem 0.6 is however quite involved (see [3]).
The following proposition establishes the relation between $\subseteq_n$ and Keisler's sandwiches. The proof is non-trivial and is skipped due to lack of space.

Proposition 0.7 *$\mathfrak{A} \subseteq_n \mathfrak{B}$ iff $\mathfrak{A} \in n$-Sand$(\mathfrak{B})$.*

References

[1] C. C. Chang and H. J. Keisler. *Model Theory*. Elsevier Science Publishers, 3 edition, 1990.

[2] Wilfrid Hodges. *A Shorter Model Theory*. Cambridge University Press, 1997.

[3] H. J. Keisler. Theory of models with generalized atomic formulas. *Journal of Symbolic Logic*, 25:1–26, 1960.