Using Preservation Theorems for Inexpressibility Results in First Order Logic

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Abstract

We investigate the applications of preservation theorems in showing inexpressibility results in FO.

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1 Introduction

Preservation theorems have been an important area of study in model theory. Preservation properties talk about collections of first order (FO) structures which are preserved under model-theoretic operations. Preservation theorems, an important class of results which study the relationship between syntax and semantics, attempt to syntactically characterize preservation properties. One of the earliest preservation theorems is the Łoś-Tarski preservation theorem which states that over arbitrary structures a FO sentence is preserved under taking substructures iff it is equivalent to a Π^0_1 sentence - a sentence which does not use any existential quantifiers [1]. In dual form, over arbitrary structures a FO sentence is preserved under extensions iff it is equivalent to a Σ^0_1 sentence - a sentence which does not use any universal quantifiers. In [2], we investigated a generalization of the "preservation under substructures" property - a property which we call as "preservation under substructures modulo finite cores" and obtained a syntactic characterization in terms of Σ^0_2 sentences for FO definable properties. We refer the reader to [2] for details. In this short article we intend to explore the applications of the above two preservation theorems and some related results in showing inexpressibility results in FO. Inexpressibility results in FO have always been of great interest since they show the limits of the expressive power of FO. These results are typically shown using Compactness, Ehrenfeucht-Fräissé games or locality arguments. It is then interesting that preservation theorems can provide a different approach to these classical results.

We clarify at the outset that the properties we consider are those of arbitrary structures (i.e. finite and infinite). We are yet to investigate how our results can be used to show inexpressibility results in the finite. Also we will be considering properties of structures of purely relational vocabularies.

2 The tools

We refer the reader to [2] for the notions and notations. We denote by $\mathbf{PS}$, all those classes of structures that are preserved under taking substructures and denote by PS the FO-definable subsets of it. We denote by $\mathbf{PSC}_f$, $\mathbf{PSC}$ and $\mathbf{PSC}(k)$, all those classes of structures that are preserved under taking substructures modulo finite cores, modulo bounded cores and modulo bounded cores of size atmost k respectively and denote their FO-definable subsets by PSC_f, PSC and $PSC(k)$ respectively. By Σ_k^0 (resp. Π_k^0), we mean sentences in PNF whose quantifier prefix begins with a $\exists$ (resp. $\forall$) and consists of $k-1$ alternations of quantifiers. We call Σ_1^0 sentences as *existential* and Π_1^0 sentences as *universal*.
We now state explicitly the results that we will use.

Theorem 1 *(Łoś-Tarski) A FO sentence ϕ is in PS iff it is equivalent to a Π_1^0 sentence.*

Theorem 2 *(Theorem 2, [2]) A sentence $\phi \in PSC_f$ iff ϕ is equivalent to a Σ_2^0 sentence.*

Lemma 1 *(Lemma 2, [2]) $PSC = PSC_f$.*

3 Inexpressibility using Theorem 1

3.1 The class of all finite structures

Suppose this class, say $\mathcal{C}$, was axiomatizable by a sentence ϕ. Observe that $\phi \in PS$ so that by Theorem 1, ϕ is equivalent to a Π_1^0 sentence. Then $\neg\phi$ is equivalent to a Σ_1^0 sentence ψ. Now ψ, being a Σ_1^0 sentence, has minimal models of finite (in fact bounded) size. Then ψ and hence $\neg\phi$ has a finite model. But this model must satisfy ϕ - contradiction.

3.2 Cycles

Suppose the class $\mathcal{C}$ of graphs containing cycles is expressible using a FO sentence ϕ. Then the models of $\neg\phi$ form the class of trees - which is preserved under substructures. Then by Theorem 1, $\neg\phi$ is equivalent to a $\forall^k$ sentence so that ϕ is equivalent to a $\exists^k$ sentence. Then ϕ forces the girth of the graph to be atmost k. Then a cycle of length $k+1$ violates ϕ but is present in $\mathcal{C}$.

3.3 Bipartiteness

A finite graph is non-bipartite iff it contains an odd length cycle. This result can be seen to be true even if the graph is infinite.

Lemma 2 *A graph G is bipartite iff G has no odd cycles.*

Proof: If G is finite, we are done. Consider the case that G is infinite. If G is bipartite, then clearly, G cannot have odd cycles. To prove the converse, suppose G is not bipartite. Construct the following set of propositional statements, $S = \{p_v \leftrightarrow \neg p_u | v, u \in V(G), (u, v) \in E(G)\}$, over the set of variables $\{p_v | v \in V(G)\}$ where $V(G)$ represents the vertex set of G and $E(G)$ represents the edge set of G. Since G is not bipartite, S is unsatisfiable. By propositional compactness, there exists a finite subset of S that is unsatisfiable. Then there is a finite subgraph of G which is not bipartite. This finite subgraph has an odd cycle. Hence, if G is not bipartite, then G has an odd cycle. ∎

Now suppose the class of bipartite graphs is definable by a sentence ϕ. Observe that $\phi \in PS$. Then by Theorem 1, ϕ is definable by a $\forall^n$ sentence so that $\neg\phi$ is definable by a $\exists^n$ sentence. Then the minimal models of $\neg\phi$ are of size bounded by n. Then consider a cycle G of length $2n + 1$. By Lemma 2, G models $\neg\phi$. But any proper induced subgraph of G is simply a collection of paths and is hence bipartite so that G is a minimal model of $\neg\phi$ - a contradiction.

4 Inexpressibility using Theorem 2 and Lemma 1

4.1 Cycles (alternate proof)

Suppose the class $\mathcal{C}$ of all graphs containing cycles is expressible by a sentence ϕ. Now in any model of ϕ, the induced subgraph formed by the vertices of any cycle serves as a core. Then $\phi \in PSC_f$. Then by Lemma 1, $\phi \in PSC$ and hence is in $PSC(k)$ for some k. Then ϕ forces the girth of the graph to be atmost k. Then a cycle of length $k + 1$ is not a model of ϕ though it is in $\mathcal{C}$.

4.2 Bipartiteness (alternate proof)

Suppose the class $\mathcal{C}$ of all bipartite graphs was expressible by a sentence ϕ. Then by Lemma 2, $\neg\phi$ captures the class $\overline{\mathcal{C}}$ of all graphs containing odd cycles. In any model of $\neg\phi$, the induced subgraph formed by the vertices of any odd cycle serves as a core. Then $\neg\phi \in PSC_f$. Then by Lemma 1, $\neg\phi \in PSC$ and hence is in $PSC(k)$ for some k. Then $\neg\phi$ forces the girth of the graph to be atmost k. Then a cycle of length $2k + 1$ is not a model of $\neg\phi$ though it is in $\overline{\mathcal{C}}$. Then $\neg\phi$, and hence ϕ, cannot exist.

We now look at two interesting applications of Theorem 2 and Lemma 1 in proving inexpressibility for which we are not aware of any way of using Theorem 1.

4.3 Caterpillars

A caterpillar is a tree in which a single finite path (the *spine*) is incident on or contains every edge. While in graph theory, caterpillars are typically finite graphs we consider this same definition in the infinite setting too - the spine is always finite in length but the “legs” can be infinite.

Suppose the class of all caterpillars is definable by a sentence ϕ. Observe that in any model of ϕ, the spine of the caterpillar acts as a finite core - then $\phi \in PSC_f$. Then by Lemma 1, $\phi \in PSC$ and hence is in $PSC(k)$ for some k. Then consider the "purely spine" caterpillar - namely a path - of length $k + 2$. There is no core in this caterpillar of size atmost k - a contradiction.
We can use the same reasoning to show that the class of "finite armies of caterpillars" where each structure is a finite collection of caterpillars is also not FO definable. Each such structure has a finite core - namely the collection of the spines of all the caterpillars but there are no bounded cores.

4.4 Graph-connectedness

We show that over the class of undirected graphs, the class of connected graphs is not FO definable. Let $\mathcal{C}$ be the class of all connected graphs. Consider $\overline{\mathcal{C}}$ - the class of all disconnected graphs. Consider a graph G in $\overline{\mathcal{C}}$. Choose two distinct connected components and choose a node from each. Check that the induced subgraph formed by these two nodes is a core in G. Then $\overline{\mathcal{C}} \in \mathsf{PSC}$ (in fact $\mathsf{PSC}(2)$). Suppose $\overline{\mathcal{C}}$ is defined by a sentence ϕ - then $\phi \in PSC$. Using Theorem 2 and Lemma 1, ϕ is equivalent to a Σ^0_2 sentence ψ. Let $\psi = \exists^m \bar{x}\ \forall^n \bar{y}\ \beta(\bar{x}, \bar{y})$. Consider a graph G_1 which is a single both-ways infinite path P. Since P is a path, by definition, any two vertices in G_1 are at a finite distance from each other so that G_1 is connected and hence in $\mathcal{C}$. Let G_2 be a graph which contains exactly 2 both-ways infinite paths (call these as P_1 and P_2). Clearly G_2 is disconnected and hence is in $\overline{\mathcal{C}}$. Then $G_2 \models \phi$ and hence $G_2 \models \psi$. Consider a witness $\bar{a}$ in G_2 for the $\bar{x}$ variables. Let $\bar{a}_1 = (a^1_1, \ldots, a^k_1)$ be the part of $\bar{a}$ that comes from path P_1 and $\bar{a}_2 = (a^1_2, \ldots, a^l_2)$ be the part of $\bar{a}$ that comes from path P_2. Note that $k + l = m$. Then in G_1, choose elements $\bar{e}$ for the $\bar{x}$ variables s.t. the elements of $\bar{e}$ are either in $\bar{e}_1$ and $\bar{e}_2$ but not both. The tuples $\bar{e}_1$ and $\bar{e}_2$ satisfy the following: (i) $\bar{e}_1 = (e^1_1, \ldots, e^k_1)$ and $\bar{e}_2 = (e^1_2, \ldots, e^l_2)$ and (ii) $d(e^i_1, e^j_1, G_1) = d(a^i_1, a^j_1, G_2)$ for $1 \leq i, j \leq k$ and $d(e^i_2, e^j_2, G_1) = d(a^i_2, a^j_2, G_2)$ for $1 \leq i, j \leq l$ and (iii) $d(e^i_1, e^j_2, G_1) > 2n$ for $1 \leq i \leq k$ and $1 \leq j \leq l$. Here $d(x, y, G)$ stands for the distance between x and y in graph G.
Now choose any $\bar{f} = (f_1, \ldots, f_n)$ as $\bar{y}$ in G_1. We intend to show that $G_1 \models \beta(\bar{e}, \bar{f})$. Towards this we importantly observe the following: It is possible to choose elements $\bar{b} = (b_1, \ldots, b_n)$ from G_2 such that the subgraph of G_1 induced by the elements of $\bar{e}$ and $\bar{f}$ is *isomorphic* to the subgraph of G_2 induced by the elements of $\bar{a}$ and $\bar{b}$ under the isomorphism H given by $H(e^i_1) = a^i_1, H(e^j_2) = a^j_2, H(f_r) = b_r$ where $1 \leq i \leq k,\ 1 \leq j \leq l$ and $1 \leq r \leq n$.
Then $G_1 \models \beta(\bar{e}, \bar{f})$ iff $G_2 \models \beta(\bar{a}, \bar{b})$. But since $\bar{a}$ is a witness for ψ in G_2, $G_2 \models \beta(\bar{a}, \bar{b})$ so that $G_1 \models \beta(\bar{e}, \bar{f})$. Since $\bar{f}$ was arbitrary, $G_1 \models \forall^n \bar{y} \beta(\bar{e}, \bar{y})$. In other words, $G_2 \models \psi$. But since ψ is equivalent to ϕ, we have $G_1 \models \phi$. But G_1 is a connected graph - a contradiction.

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References

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