Clique-Width of Point Configurations

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Abstract. While structural width parameters (of the input) belong to the standard toolbox of graph algorithms, it is not the usual case in computational geometry. As a case study we propose a natural extension of the structural graph parameter of *clique-width* to geometric point configurations represented by their *order type*. We study basic properties of this clique-width notion, and relate it to the monadic second-order logic of point configurations. As an application, we provide several linear FPT time algorithms for geometric point problems which are NP-hard in general, in the special case that the input point set is of bounded clique-width and the clique-width expression is also given.

Keywords: Point configuration · Order type · Fixed-parameter tractability · Relational structure · Clique-width

1 Introduction

An order type is a useful means to characterize the combinatorial properties of a finite point configuration in the plane. As introduced in Goodman and Pollack [17,18], the *order type* of a given set P of points assigns, to each ordered triple $(a, b, c) \in P^3$ of points, the orientation (either clockwise or counter-clockwise) of the triangle abc in the plane. More generally, if the point set P is not in a general position, the triple (a, b, c) may also be collinear (as the natural third option).

Knowing the order type of a point set P is sufficient to determine some useful combinatorial properties of the geometric set P, such as the convex hull of P and other. For example, problems of finding convex holes in P or dealing with the intersection pattern of straight line segments with ends in P, can be solved by looking only at the order type of P and not on its geometric properties. That is

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why order types of points sets are commonly studied from various perspectives in the field of computational geometry, e.g., [2–6,16,19,27].

On the other hand, knowing the order type of P is obviously not sufficient to answer questions involving truly "geometric" aspects of P, e.g., distances in P (straight-line or geodesic), or angles between the lines or the area of polygons within P. Nevertheless, even in such geometry-based problems, a more efficient subroutine computing with the order type of P might speed-up the overall computation, which can be a promising direction for future research.

Unlike in the area of graphs and graph algorithms, where structural width parameters are very common for many years, at least since the 90's, no similar effort can be seen in combinatorial and computational geometry. We would like to introduce, in this paper, possible combinatorial handling of "structural complexity" of a given point configuration P through defining its "width" (which we would assume to be small for the studied inputs).

Inspired by graph structure parameters, the obvious first attempt could be to extend the traditional notion of *tree-width* [26]. Such an extension is technically possible (cf. tree-width of the Gaifman graph of a relational structure), but the huge problem is that for the tree-width to be upper-bounded, the underlying structure must be "sparse" – in particular, it can only have a linear number of edges/tuples. This is clearly not satisfied for the order type in which about half of all triples are of each orientation.

A better option comes with another traditional, but not so well-known, notion of *clique-width* [12]. Clique-width can be bounded even on dense graphs, such as on cliques, and, similarly to the case of Courcelle's theorem [9] for tree-width, clique-width also enjoys some nice metaalgorithmic properties, e.g. [11,15]. This includes solving any decision (and some optimization as well) problems formulated in the monadic second-order (MSO) logic in linear time. Hence, alongside the (Sect. 2) proposed definition of the clique-width of point configurations, we will introduce the MSO language of their order types and discuss which problems can be formulated in this language (and hence solved in linear time if a point set with a decomposition of bounded clique-width is given on the input).

Arguments which were omitted due to space restrictions can be found in the full paper preprint arXiv:2004.02282.

2 Order Types and Clique-Width

We now recall the notion of an order type in a formal setting, and propose a definition of the clique-width of (the order type of) a point configuration, based on a natural specialization of the very general concept of clique-width of relational structures. A *relational structure* $S = (U, R_1^S, \ldots, R_a^S)$ of the signature $\sigma = \{R_1, \ldots, R_a\}$ consists of a universe (a finite set) U and a (finite) list of relations $R_1^S, \ldots, R_a^S$ over U. For instance, for graphs, $U = V(G)$ is the vertex set and $R_1^G = E(G)$ is the binary symmetric relation of edges of G.

For a set of points P, here *always* considered in the plane, consider a map $\omega : P^3 \to \{+,-,0\}$ where $\omega(a,b,c) = 0$ if the triple of points[1] a,b,c is collinear, $\omega(a,b,c) = +$ if abc forms a counter-clockwise oriented triangle, and $\omega(a,b,c) = -$ otherwise. Then ω is traditionally called the order type of P, but we, for technical reasons, prefer defining the *order type* of P as the ternary relation $\Omega \subseteq P^3$ such that $(a,b,c) \in \Omega$ iff $\omega(a,b,c) = +$. Hence we have formally got a relational structure (P,Ω) of the signature consisting of one ternary symbol. We will also write $\Omega(P)$ to emphasize that Ω is the order type of the point set P.

Observe that $\omega(a,b,c) = -$ iff $(b,a,c) \in \Omega$, and $\omega(a,b,c) = 0$ iff $(a,b,c) \notin \Omega$ and $(b,a,c) \notin \Omega$. Hence, the relation Ω fully determines the usual order type of P. Furthermore, $(a,b,c) \in \Omega$ implies $(b,c,a) \in \Omega$ and $(c,a,b) \in \Omega$, and so we call the set of triples $\{(a,b,c),\ (b,c,a),\ (c,a,b)\}$ the *cyclic closure* of $(a,b,c) \in \Omega$.

Unary Clique-Width. We start with the definition of ordinary graph clique-width. Let an *ℓ-expression* be an algebraic expression using the following four operations on vertex-labelled graphs using ℓ labels:

(u1) create a new vertex with single label i;
(u2) take the disjoint union of two labelled graphs;
(u3) add all edges between the vertices of label i and label j $(i \neq j)$; and
(u4) relabel all vertices with label i to label j.

The *clique-width* $\mathrm{cw}(G)$ of a graph G equals the minimum ℓ such that (some labelling of) G is the value of an ℓ-expression.

The idea behind this definition is that the edge set of a graph G can be constructed with "bounded amount of information"; this is since we have only a fixed number of distinct labels and vertices of the same label are, intuitively speaking, further indistinguishable by the expression.

This definition has an immediate generalization to the *unary clique-width* of an order type $\Omega(P)$ of a point set P (the adjective referring to the fact that labels occur as unary predicates in the definition): replace (u3) with

(u3') add to Ω the cyclic closures of all triples (a,b,c) of distinct elements such that a is labelled i, b is labelled j and c is labelled k.

Unfortunately, although being very simple, this definition is generally not satisfactory due to problems discussed, e.g., in [1] and specifically illustrated for order types in our Proposition 2.

Multi-ary Clique-Width. While in the case of graphs (whose edge relation is binary) it is sufficient to consider clique-width expressions with unary labels, for the ternary order-type relation (as well as for other relational structures of higher arity) it is generally necessary to allow creation of "intermediate" binary labels, which are labelled pairs of points of P.

This generalization, which is in agreement with the treatment by Blumensath and Courcelle [7], leads to the proposed new definition:

[1] Note that if any two of a,b,c are not distinct, then we automatically get $\omega(a,b,c) = 0$.

Definition 1 (Clique-width of a point configuration). Consider an algebraic expression $\mathcal{E}$ using the following five operations on labelled relational structures (of arity 3 in this case) over point sets:

(w1) create a new point with single label i;
(w2) take the disjoint union of two point sets;
(w3) for every two points, point a of label i and point b of label j ($i \neq j$), give the ordered pair (a, b) binary label k;[2]
(w4) for every three pairwise distinct points, a, b and c such that c is of (unary) label i, and the pair (a, b) is of (binary) label k, add to the structure the cyclic closure of the ordered triple (a, b, c);
(w4') under the same conditions as in (w4), add the cyclic closure of (b, a, c);
(w5) relabel all tuples (singletons or pairs) with label i to label j of equal arity.

The *value* of such expression $\mathcal{E}$ is the ternary relational structure on the points created by (w1) and consisting of the triples added by (w4) and (w4'). The auxiliary labels introduced in $\mathcal{E}$ are no longer relevant after the evaluation of $\mathcal{E}$.

The *width* of an expression $\mathcal{E}$ constructed as in (w1)–(w5) equals the sum of arities of the labels occuring in $\mathcal{E}$.[3] The *clique-width* $cw(P)$ of a point configuration P equals the minimum ℓ such that the order type $\Omega(P)$ of P is the value of an expression of width at most ℓ.

Note that, although the clique-width is a concrete natural number, we will not be interested in the exact value of it, but instead study whether the clique-width is bounded or unbounded on a given class of point configurations.

For a closer explanation of this concept, we present a basic example:

Proposition 2. *Let P be an arbitrary finite set of points in a strictly convex position.*[4] *Then the clique-width of P is bounded by a constant, while the unary clique-width of P is unbounded.*

Proof Outline. Let the points of P be $p_1, p_2, \ldots, p_n$ in the counter-clockwise order (starting arbitrarily). We start with p_1 and stepwise add p_2, p_3 etc., changing previous points to label 1 and the added point created with unique label 2. See Fig. 1. Along the steps, after the creation of p_j, we add the binary label 3 to all pairs labelled 1 and 2, i.e., to (p_i, p_j) for all $i < j$, and create the order triples $(p_i, p_{i'}, p_j)$ of three distinct points over all pairs $(p_i, p_{i'})$ of label 3 and p_j of label 2. This construction witnesses that the clique-width of P is at most 4.

On the other hand, take unary clique-width with ℓ labels, and $|P| \geq 2\ell + 1$. An arbitrary ℓ-expression for $\Omega(P)$ must involve a union operation (the "last" one) over two sets such that one has more than ℓ points. Then, at the time of taking the union, there are two points a, b of the same label in the set by the pigeon-hole principle. Let c be any point from the other set. Then there is no

[2] After this operation, (a, b) may hold more than one binary label, which is ok.
[3] Note that this 'sum of arities' measure directly generalizes the number ℓ of unary labels in the expression of (u1)–(u4).
[4] That is, in the convex hull of P every point of P is a vertex.

way, based on the labels, to distinguish between the triples (a, b, c) and (b, a, c), which must have the opposite orientations in $\Omega(P)$. Therefore, the clique-with of P must be at least $\ell + 1$.

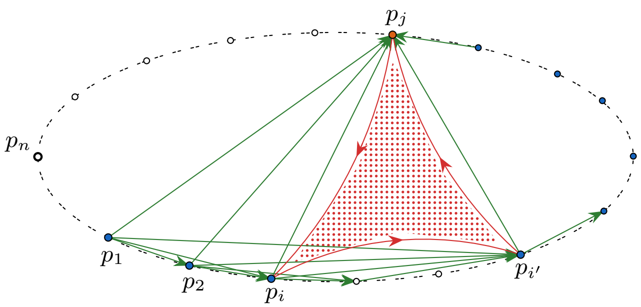

Fig. 1. An illustration of the expression (width 4) in Proposition 2. Unary labels 1 are blue (on $p_1, \ldots, p_{j-1}$), the unique label 2 is orange (on p_j just added), and the binary labels 3 are with green arrows. We are just creating the red triple(s) $(p_i, p_{i'}, p_j)$. (Color figure online)

Annotated Point Configurations. In some situations, it may be useful to consider a point configuration P with additional information (or structure) on the points or selected pairs of them. An exemplary use case for such annotations is to study polygons, with P as the vertex set, for which case we are considering an order type $\Omega(P)$ together with a directed Hamiltonian cycle on P representing the counter-clockwise boundary of P.

Formally, we simply consider relational structures (over P) with the signature consisting of the ternary order type and arbitrary binary or unary symbols. The *clique-width* of such an *annotated point configuration* P is, naturally, as in Definition 1 with additional rules that some of the auxiliary unary and binary labels are at the end turned into the desired unary and binary relations on P.

3 MSO Logic of Order Types

The beginning of this section is devoted to a short introduction of the *monadic second-order (MSO) logic* of relational structures. Recall a *relational structure* $S = (U,\ R_1^S, \ldots, R_q^S)$ of the signature $\sigma = \{R_1, \ldots, R_q\}$.

The language of MSO logic (of the signature σ) then consists of the standard propositional logic, quantifiers $\forall, \exists$ ranging over elements and subsets of the universe U, and the relational symbols $R_1, \ldots, R_q$ with the following meaning: for R_i of arity a, we have $S \models R_i(x_1, \ldots, x_a)$ if and only if $(x_1, \ldots, x_a) \in R_i^S$.

In our specific case of order types $\Omega(P)$ of point sets P, we use the relational symbol $ordccw(x_1, x_2, x_3)$ for Ω within MSO logic. For example, we can express that a point y lies strictly in the convex hull of points x_1, x_2, x_3 as follows

$$\left[ordccw(x_1,x_2,x_3) \wedge \bigwedge\nolimits_{i=1,2,3} ordccw(x_i,x_{i+1},y)\right] \vee \qquad (1)$$
$$\left[ordccw(x_3,x_2,x_1) \wedge \bigwedge\nolimits_{i=1,2,3} ordccw(x_{i+1},x_i,y)\right],$$

where x_4 is taken as x_1.

More generally, we can express that a point $y \in P$ belongs to the convex hull (not necessarily strictly now) of a set $X \subset P$ with the following formula:

$$\begin{aligned} convhull(X,y) \;\equiv\; & y \in X \;\vee\; \forall x,x' \in X \\ & \left[\left(x \neq x' \wedge \forall z \in X \neg\, ordccw(x',x,z)\right) \rightarrow \neg\, ordccw(x',x,y)\right] \end{aligned} \qquad (2)$$

Then we may express, for example, that a set $X \subseteq P$ is a convex hole (i.e., no point outside of X belongs to the convex hull of X, and no point of X belongs to the convex hull of the rest of X) with the following:

$$\forall y \notin X\, (\neg\, convhull(X,y)) \;\wedge\; \forall Y \subseteq X \forall z \in X (convhull(Y,z) \rightarrow z \in Y) \qquad (3)$$

Further similar examples are easy to come up with.

Interpretations and Transductions. We sketch the concept of "translating" between relational structures. Consider relational signatures $\sigma = \{R_1,\ldots,R_q\}$ and $\tau = \{R'_1,\ldots,R'_t\}$. A (simple) *MSO interpretation of τ-structures in σ-structures* is a t-tuple of MSO formulas $\Psi = (\psi_i : 1 \leq i \leq t)$ of the signature σ, where the number of free variables of ψ_i equals the arity a_i of R'_i. A τ-structure T is *interpreted* in a σ-structure S via Ψ if T and S share the same ground set U and, for each $1 \leq i \leq t$, we have $(x_1,\ldots,x_{a_i}) \in R'^{T}_i \iff S \models \psi_i(x_1,\ldots,x_{a_i})$.

As a short example, consider a point set P and its mirror image P'. Then the order type $\Omega(P')$ can be interpreted in $\Omega(P)$ simply by taking $\psi_1(a,b,c) \equiv ordccw(b,a,c)$. The true power of interpretations will show up in the following.

There is a more general concept of a *transduction* from a σ-structure S to a set of τ-structures which, before taking an (MSO) interpretation, has abilities (in this order of application); (i) to equip S with a fixed number of arbitrary parameters given as unary labels (because of this, the result of a transduction is not deterministic, but a set of τ-structures), (ii) to "amplify" the ground set of S by taking a bounded number of disjoint copies of S, and (iii) to subsequently restrict the ground set by an MSO formula with one free variable. See Courcelle and Engelfriet [10] for more technical details on transductions.

Considering a transduction Ψ (as described above) and a σ-structure S, let $\Psi(S)$ denote the set of τ-structures which result from S under the transduction Ψ. For a class of relational structures $\mathcal{S}$, the image under a transduction Ψ of the class $\mathcal{S}$ is the union of all transduction results, precisely, $\Psi(\mathcal{S}) := \bigcup_{S\in\mathcal{S}} \Psi(S)$.

Note that one can come up with various notions of clique-width of relational structures (also giving distinct numbers for the same structure), but the underlying essence is always "the same". In order to smoothen out marginal technical differences between the various definitions, we consider the following. We say that a class $\mathcal{S}$ is of *bounded clique-width* if there exists a constant h such that the clique-width of every $S \in \mathcal{S}$ is at most h. On such abstract level, we then have the following *crucial characterization* (essentially a metadefinition):

Theorem 3 (Blumensath and Courcelle [7, Proposition 27]). *A class $\mathcal{S}$ of finite relational structures (of the same signature) is of bounded clique-width, if and only if $\mathcal{S}$ is contained in the image of the class of finite trees under an MSO transduction.*

For a *very informal* explanation of the meaning of this statement, we remark that a tree which is the preimage of the mentioned transduction gives a hierarchical structure to the clique-width expression in Definition 1. The arbitrary transduction parameters then determine particular operations (and labelling) used within the expression, and the formula(s) of a final interpretation roughly encodes Definition 1 itself. No copying ("amplification") is necessary there.

Since the concept of a transduction is transitive, Theorem 3 implies:

Corollary 4. *If a class $\mathcal{S}$ of order types (of points) is of bounded clique-width, then the image of $\mathcal{S}$ under an MSO transduction is also of bounded clique-width.*

Deciding MSO Properties. Perhaps the most important application of bounded clique-width of point configurations P could be in faster deciding of MSO-definable properties (and, in greater generality, of some optimization and counting properties as well, see examples in [11]) of the order type of P.

Theorem 5 (Courcelle et al. [11], via Theorem 3). *Consider a class $\mathcal{S}$ of finite relational structures of signature σ and of bounded clique-width. For any MSO sentence φ of signature σ, if a structure $S \in \mathcal{S}$ is given on the input alongside with a clique-width expression of bounded width, then we can decide in linear time whether $S \models \varphi$ (i.e., whether S has the property φ).*

Furthermore, under the same assumptions for $\mathcal{S}$ and for an MSO formula $\varphi(X)$ with a free set variable X, we can find in linear time a minimum- or maximum-cardinality set X such that $S \models \varphi(X)$, and we can enumerate all sets X such that $S \models \varphi(X)$ in time which is linear in the input plus output size.

4 Assorted Examples

First, to give readers a better feeling about how big the clique-width of "nicely looking" point sets in the plane can be, we show the following:

Theorem 6. *Let P be a point configuration, $P_0 \subseteq P$ and $d = |P \setminus P_0|$.*

(a) If all points of P_0 are collinear, then the clique-width of P is in $\mathcal{O}(d)$.
(b) Assume the points of P_0 are in a strictly convex position. If $d \leq 1$, then the clique-width of P is bounded (by a constant). On the other hand, there exist examples already with $d = 2$ and unbounded clique-width of P.

Proof Outline. In case (a), we first create the d points of $P \setminus P_0$, each with its unique label, and their counter-clockwise order triples. See Fig. 2(a). Then we stepwise create the collinear points of P_0, ordered from left to right. During the steps, we add binary labels on P_0 between each pair from left to right, and we also in the right order create the needed order triples having one point in P_0 and two points in $P \setminus P_0$. At the end, we easily create from the binary labels on P_0 the remaining order triples having two points in P_0 and one in $P \setminus P_0$.

In case (b), if $d = 1$, we construct an expression similarly as in Proposition 2, but we simultaneously proceed in two subsequences of the counter-clockwise perimeter of P_0, "opposite" to each other. This process of construction allows us to create also the counter-clockwise order triples involving the sole point of $P \setminus P_0$ (in "the middle").

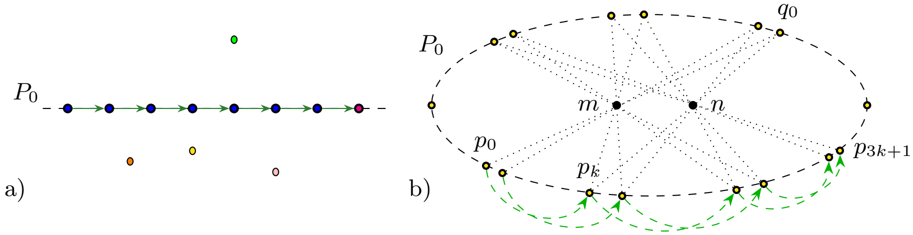

Fig. 2. Illustrations of the two parts of Theorem 6. (a) Labelling for an expression of bounded width. (b) A sketch of interpreting a large grid within the point configuration. (Color figure online)

In case (b) with $d \geq 2$, we present a construction informally shown in Fig. 2(b). The underlying idea is to use the points of $P \setminus P_0$ to "mutually relate" opposite points of P_0, such as the depicted collinear triples p_0, m, q_0 and q_0, n, p_k. Collinear triples are easy to detect within the order type, hence we can this way interpret the binary relation between p_0 and p_k, analogously between subsequent p_1 and p_{k+1}, and so on (see the green dashed arrows in the picture). Together with a description of neighbouring points of P_0 – see x and x' in (2), we can interpret an arbitrarily large square grid graph on the points $p_0, p_1, \ldots, p_k, p_{k+1}, \ldots$ Since the square grid is a folklore basic example of unbounded clique-width [10], Corollary 4 implies that the clique-width of such configurations P (with $d = 2$) is unbounded. A similar construction, albeit more complicated, with "doubling" the points q_i, can show the same result without having collinear triples in P.

Some NP-Hard Problems of Point Configurations

As already mentioned, perhaps the most interesting computing application of clique-width of point sets could be in designing algorithms which run in parameterized polynomial, or even linear, time with respect to the clique-width as the parameter. This is especially relevant for problems for which no such algorithms are believed to exist in general, such as for NP-hard problems.

A parameterized problem has an *FPT algorithm* if the algorithm runs in time $\mathcal{O}(f(d) \cdot n^c)$ where f is an arbitrary computable function of the (fixed) parameter d, and c is a constant. If $c = 1$, then we speak about a linear FPT algorithm (e.g., this is the complete case of Theorem 5).

Since, except the binary case such as that of graphs, there is no known FPT algorithm (even approximation one) for finding a clique-width expression of relational structures of bounded clique-width, we must assume that an expression of bounded width is given alongside with the input point configuration. Notice that for the above presented examples of small clique-width, the relevant expressions are very natural and easy to come with.

General Position Subset. This problem asks whether, for a given point set P and integer k, there exists a subset $Q \subseteq P$ such that no three points of Q are collinear and $|Q| \geq k$. This problem is NP-hard and APX-hard by [14].

Theorem 7. *Assume a point set P is given alongside with a clique-width expression (for $\Omega(P)$) of width d. Then the* General position subset *problem of P is solvable in linear FPT time with respect to the parameter d.*

Proof. We write the MSO formula

$$\varphi(X) \equiv \forall x, y, z \in X \left[x \neq y \neq z \neq x \rightarrow \left(\mathit{ordccw}(x, y, z) \vee \mathit{ordccw}(y, x, z) \right) \right]$$

to say that no three points in X are collinear, and then compute using Theorem 5 the value $\max_{\Omega(P) \models \varphi(X)} |X|$ and compare to k. □

A very similar simple approach works also for the NP-hard problem Hitting set for induced lines [25], which asks for a minimum-cardinality subset $H \subseteq P$ such that the lines between each pair of points of P all contain a point of H.

Minimum Convex Partition. Consider a given point set P and an integer k. The objective of this problem [13] is to decide whether the convex hull $\mathit{conv}(P)$ of P can be partitioned into $\leq k$ convex faces. By a *convex face* in this situation we mean the convex hull of a subset $Q \subseteq P$ which is a convex hole of P (recall (3)). Note that in our definition Q must be strictly convex, but we may as well apply a non-strict variant in which some points of Q (possibly) are not vertices of $\mathit{conv}(Q)$ but lie on the boundary of $\mathit{conv}(Q)$; the arguments would be similar.

This problem has been recently claimed NP-hard [20]. Unfortunately, inherent limitations of MSO logic do not allow us to directly formulate the Minimum convex partition as an MSO optimization problem (one is not allowed to quantify set families), but we can handle it if we take k as an additional parameter.

Theorem 8. *Assume a point set P given alongside with a clique-width expression of width d. The* Minimum convex partition *problem of P into $\leq k$ convex faces is solvable in linear FPT time with respect to the parameter $d + k$.*

Proof Outline. Let $convhole(X)$ denote the MSO formula (3). We may now write

$$\exists X_1, \ldots, X_k \left[\bigwedge_{1 \le i \le k} convhole(X_i) \ \wedge \ convpartition(X_1, \ldots, X_k) \right]$$

where the subformula *convpartition* checks whether the convex hulls of the sets X_i partition $conv(P)$. At this point, we know that each X_i is a convex hole in P, and we further test for set inclusion and the following two conditions:

- the boundaries of $conv(X_i)$ and $conv(X_j)$ $(1 \le i < j \le k)$ do not cross, and
- every boundary edge of $conv(X_i)$ is, at the same time, a boundary edge of exactly one of $conv(X_j)$ $(i \ne j)$ or of $conv(P)$.

Both conditions can be, although not easily, stated in MSO over order types.

Terrain Guarding. Another NP-hard problem formulated on point sets [22] is that of guarding an x-monotone polygonal line L with the given vertex set P. The objective of guarding is to find a minimum-cardinality vertex guard set $G \subseteq P$ such that every point ℓ on L is seen by some point $g \in G$ "from above the terrain", that is, the straight line segment from g to ℓ is never strictly below L.

Note that the point set P (no two points of the same x-coordinate) uniquely determines the terrain L, with the vertices ordered by their x-coordinates as $P = (p_1, p_2, \ldots, p_n)$. However, the order type $\Omega(P)$ does not (unless we would add an auxiliary point "at infinity" in the y-axis direction). That is why we assume the terrain L given as a relational structure consisting of ternary $\Omega(P)$ and the binary successor relation consisting of the pairs (p_1, p_{i+1}) for $1 \le i < n$.

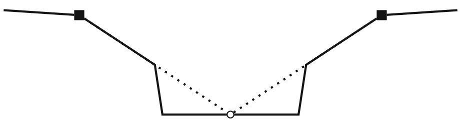

Fig. 3. Guarding a terrain: the two black square vertices guard the whole terrain, but the bottom horizontal segment is not seen by any single one of them. To turn this (pair of guards) into a valid segmented terrain guarding instance with a solution, we may subdivide the bottom segment into two segments with a new vertex (the hollow dot) of the terrain – each guard would then see an entire one of the two segments.

There is one further complication in regard of the order type $\Omega(P)$ of the terrain in this problem: if, in an instance, some edge of L is seen together by two guards, but no one sees the full edge, then knowing only $\Omega(P)$ is not sufficient to verify validity of such a solution (see Fig. 3). That is why we define here the Segmented terrain guarding variant as follows: for every segment s of L there must exist a vertex guard g seeing entire s and, moreover, there is a dedicated subset $P_1 \subseteq P$ such that the guards g are selected from P_1. By a natural subdivision of terrains in the hard instances of terrain guarding [22] we immediately get that also Segmented terrain guarding is NP-hard.

Theorem 9. *Assume a polygonal terrain L given alongside with a clique-width expression of width d (defining both the successor relation and the order type of the vertices, cf. end of Sect. 2). The* Segmented terrain guarding *problem of L is solvable in linear FPT time with respect to the parameter d.*

Proof Outline. We show a formula *seguard*(X) stating that every segment of the terrain L is seen by one point of X. Then, we verify that, for every successive pair of vertices (p_i, p_{i+1}) of L, there exists $x \in X$ such that;

– the triple (x, p_i, p_{i+1}) is oriented counter-clockwise (for x to see the segment $\overline{p_i p_{i+1}}$ "from above"), and
– no "peak" z on L between $\overline{p_i, p_{i+1}}$ and x is oriented clockwise from (x, p_{i+1}) (if z is to the left of x) or counter-clockwise from (x, p_i) (z to the right of x).

This suffices since L is x-monotone. Then Theorem 5 finishes the argument.

We can similarly handle the orthogonal terrain guarding problem which is also NP-hard [8]. Another possible extension is to minimize the sum of weighted guards, using a weighted variant of Theorem 5 (as in [11]). However, our approach to terrain guarding cannot be directly extended to the traditional and more general *Art gallery* (guarding) problem [23], not even in the adjusted case when each edge of the polygon is seen by a single vertex guard. This is due to possible presence of "blind spots" in the interior of the polygon which cannot be determined knowing just the order type $\Omega(P)$ and the boundary edges of the polygon on P. Interested readers may find more in the full paper.

Polygon Visibility Graph. As we have mentioned the Art gallery problem, we briefly add that people are also studying problems related to the *visibility graph* of a given polygon Q. The visibility graph of Q has the same vertex set as Q and the edges are those line segments with ends in the vertices of Q which are disjoint from the complement of the polygon. We give the following toolbox:

Theorem 10. *Assume a polygon Q with vertex set P given as a relational structure consisting of the order type $\Omega(P)$ and the counter-clockwise Hamiltonian cycle of edges of Q. Then the visibility graph of Q has an MSO interpretation in Q.*

5 Conclusions

We managed to show, in this limited space, only few example applications of bounding the clique-width in efficient parameterized algorithms for geometric point problems. More examples of similar kind could be added but, as a future work, we would especially like to investigate possible applications to "metric" problems. Of course, MSO logic of order types cannot express metric properties of a point set, but it could be possible that in some problems the enumerative part of Theorem 5 provided us with a relatively short list of small subconfigurations which would then be processed even by brute force, resulting in a faster algorithm. For instance, we suggest to investigate in this manner the problem of a minimum area triangle on a given point set, which is in general 3SUM-hard (that is, not believed to have a subquadratic algorithm).

Another possible extension would be to consider order types in dimension 3 (or higher), but then even a strictly convex point set could easily have unbounded clique-width – the quaternary relational structures of such order types just seem to be too complex even in very simple cases.

Lastly, we mention another very natural question; can the clique-width of a point configuration be at least approximated by an FPT algorithm with the width as the fixed parameter? Such an approximation is possible in the case of graph clique-width [21,24], thanks to the close relation of graph clique-width to rank-width and to binary matroids. Perhaps the natural correspondence of order types to oriented matroids could be of some help in this research direction.

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References

1. Adler, H., Adler, I.: A note on clique-width and tree-width for structures. CoRR abs/0806.0103 (2008). http://arxiv.org/abs/0806.0103
2. Aichholzer, O., Aurenhammer, F., Krasser, H.: Enumerating order types for small point sets with applications. Order **19**(3), 265–281 (2002)
3. Aichholzer, O., et al.: Minimal representations of order types by geometric graphs. In: Archambault, D., Tóth, C.D. (eds.) GD 2019. LNCS, vol. 11904, pp. 101–113. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-35802-0_8
4. Aichholzer, O., Krasser, H.: Abstract order type extension and new results on the rectilinear crossing number. Comput. Geom. **36**(1), 2–15 (2007)
5. Aichholzer, O., Kusters, V., Mulzer, W., Pilz, A., Wettstein, M.: An optimal algorithm for reconstructing point set order types from radial orderings. Int. J. Comput. Geom. Appl. **27**(1–2), 57–84 (2017)
6. Aloupis, G., Iacono, J., Langerman, S., Özkan, Ö., Wuhrer, S.: The complexity of order type isomorphism. In: SODA, pp. 405–415. SIAM (2014)
7. Blumensath, A., Courcelle, B.: Recognizability, hypergraph operations, and logical types. Inf. Comput. **204**(6), 853–919 (2006)
8. Bonnet, É., Giannopoulos, P.: Orthogonal terrain guarding is NP-complete. JoCG **10**(2), 21–44 (2019)
9. Courcelle, B.: The monadic second order logic of graphs I: recognizable sets of finite graphs. Inf. Comput. **85**, 12–75 (1990)
10. Courcelle, B., Engelfriet, J.: Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach, Encyclopedia of Mathematics and Its Applications, vol. 138. Cambridge University Press, Cambridge (2012)
11. Courcelle, B., Makowsky, J.A., Rotics, U.: Linear time solvable optimization problems on graphs of bounded clique-width. Theory Comput. Syst. **33**(2), 125–150 (2000)
12. Courcelle, B., Engelfriet, J., Rozenberg, G.: Context-free handle-rewriting hypergraph grammars. In: Ehrig, H., Kreowski, H.-J., Rozenberg, G. (eds.) Graph Grammars 1990. LNCS, vol. 532, pp. 253–268. Springer, Heidelberg (1991). https://doi.org/10.1007/BFb0017394
13. Demaine, E., Fekete, S., Keldenich, P., Krupke, D., Mitchell, J.: Geometric optimization challenge, part of CG Week in Zurich, Switzerland, 22–26 June 2020 (2020). https://cgshop.ibr.cs.tu-bs.de/competition/cg-shop-2020/
14. Froese, V., Kanj, I.A., Nichterlein, A., Niedermeier, R.: Finding points in general position. Int. J. Comput. Geom. Appl. **27**(4), 277–296 (2017)

15. Ganian, R., Hliněný, P.: On parse trees and Myhill-Nerode-type tools for handling graphs of bounded rank-width. Discrete Appl. Math. **158**(7), 851–867 (2010)
16. Goaoc, X., Hubard, A., de Joannis de Verclos, R., Sereni, J., Volec, J.: Limits of order types. In: Symposium on Computational Geometry. LIPIcs, vol. 34, pp. 300–314. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2015)
17. Goodman, J.E., Pollack, R.: Multidimensional sorting. SIAM J. Comput. **12**(3), 484–507 (1983)
18. Goodman, J.E., Pollack, R.: Upper bounds for configurations and polytopes in R^d. Discrete Comput. Geom. **1**, 219–227 (1986)
19. Goodman, J.E., Pollack, R., Sturmfels, B.: Coordinate representation of order types requires exponential storage. In: Proceedings of the 21st Annual ACM Symposium on Theory of Computing, pp. 405–410. ACM (1989)
20. Grelier, N.: Minimum convex partition of point sets is NP-hard. CoRR abs/1911.07697 (2019). http://arxiv.org/abs/1911.07697
21. Hliněný, P., Oum, S.: Finding branch-decomposition and rank-decomposition. SIAM J. Comput. **38**, 1012–1032 (2008)
22. King, J., Krohn, E.: Terrain guarding is NP-hard. SIAM J. Comput. **40**(5), 1316–1339 (2011)
23. Lee, D.T., Lin, A.K.: Computational complexity of art gallery problems. IEEE Trans. Inf. Theory **32**(2), 276–282 (1986)
24. Oum, S., Seymour, P.D.: Approximating clique-width and branch-width. J. Comb. Theory Ser. B **96**(4), 514–528 (2006)
25. Rajgopal, N., Ashok, P., Govindarajan, S., Khopkar, A., Misra, N.: Hitting and piercing rectangles induced by a point set. In: Du, D.-Z., Zhang, G. (eds.) COCOON 2013. LNCS, vol. 7936, pp. 221–232. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-38768-5_21
26. Robertson, N., Seymour, P.D.: Graph minors. II. Algorithmic aspects of tree-width. J. Algorithms **7**(3), 309–322 (1986)
27. Roy, B.: Point visibility graph recognition is NP-hard. Int. J. Comput. Geom. Appl. **26**(1), 1–32 (2016)