A Summary of Ph.D. Dissertation

Abhisekh Sankaran

Title of dissertation:	A generalization of the Łoś-Tarski preservation theorem
Advisors:	Prof. Bharat Adsul and Prof. Supratik Chakraborty
Examiners:	Prof. Mihir Kumar Chakraborty and Prof. Anuj Dawar
Submitted to:	The Department of Computer Science and Engineering, Indian Institute of Technology Bombay, India.
Submitted on:	10 August, 2016
Pages:	180 pp.

Summary:

Classical model theory is a subject within mathematical logic, that seeks to study mathematical structures for their properties that can be expressed in formal languages like first order logic. This dissertation is about a generalization of a very well-known result from classical model theory, called the *Łoś-Tarski preservation theorem.* The theorem, proved by the logicians Jerzy Łoś and Alfred Tarski in the mid 1950s, provides a syntactic characterization, over the class of all structures finite and infinite, of a widely studied notion in mathematical literature, called *preservation under substructures*, also known as *hereditariness.* We present a parameterized generalization of the Łoś-Tarski theorem by providing a syntactic characterization over all structures, of a parameterized generalization of preservation under substructures, that we call *preservation under substructures modulo k-cruxes*, where k is a natural number. The Łoś-Tarski theorem is a special case of our generalization when k equals 0. We call our result the *generalized Łoś-Tarski preservation theorem*, abbreviated $\mathsf{GLT}(k)$.

While $\mathsf{GLT}(k)$ syntactically characterizes a semantic property that generalizes hereditariness, the same result, flipped around, can be seen to provide new semantic characterizations of the Σ^0_2 and Π^0_2 syntactic classes of first order logic sentences, which are sentences whose syntactic structure consists of two blocks of quantifiers followed by a quantifier-free formula. These classes of first order logic sentences have not only been of much interest in mathematics, but have also proven to be extremely important to computer science: the Σ^0_2 class (also called *effectively propositional logic*) has been particularly useful to the program verification and program synthesis communities, while the Π^0_2 class has been of marked relevance to the database community in the context of data exchange, data integration, data interoperability, and query answering over RDF and OWL knowledge. To the best of our awareness, $\mathsf{GLT}(k)$ is the first to relate *counts of quantifiers* appearing in Σ^0_2 and Π^0_2 sentences to natural quantitative properties of models, and thereby provides finer characterizations of the Σ^0_2 and Π^0_2 classes than all existing results in the classical model theory literature that characterize these classes. Further, in contrast to the characterizing notions contained in the literature results just mentioned, our notion of preservation under substructures modulo k-cruxes is *finitary and combinatorial* in nature, and stays non-trivial over finite structures as well.

The subject of *finite model theory* seeks to study the logical properties of finite structures, specifically, structures of interest in computer science, and one of the current and active themes in this subject is the investigation of classes of *finite* structures over

which the Łoś-Tarski theorem holds. We study $\mathsf{GLT}(k)$ in this context, and formulate an abstract combinatorial property of finite structures, which when satisfied by a class of finite structures, ensures that $\mathsf{GLT}(k)$ holds over the class. This property, that we call the *Equivalent Bounded Substructure Property*, abbreviated EBSP, intuitively states that a large structure contains a small "logically similar" substructure. It turns out that this simply stated property is enjoyed by a variety of classes of interest in computer science. Examples include classes defined using posets, such as words, trees (unordered, ordered or ranked) and nested words, and classes of graphs, such as cographs, graph classes of bounded tree-depth, those of bounded shrub-depth, m-partite cographs, and more generally, graph classes that are well-quasi-ordered under the (isomorphic) embedding relation. All of these classes have attracted significant attention for their excellent logical and algorithmic properties, and moreover, many of these classes are very recently defined (in the last 10 years). We go on further to give a variety of methods to construct new classes satisfying EBSP from those known to satisfy this property. Specifically, we show that EBSP remains preserved under finite intersections, finite unions and under taking hereditary subclasses, and also remains preserved under several well-studied operations on structures, such as complementation, transpose, the line graph operation, disjoint union, join, and various products including the cartesian, tensor, lexicographic and strong products. This enables constructing a wide spectrum of classes that satisfy EBSP, and hence $\mathsf{GLT}(k)$, and hence the Łoś-Tarski theorem.

We find it worth mentioning of EBSP, that it has a remarkably close resemblance to a very important property studied in the classical model theory literature, called the *downward Löwenheim-Skolem property*; indeed EBSP can be well regarded as a *finitary analogue* of the latter (which is intrinsically "infinitary" in nature). It is pleasantly surprising that while the downward Löwenheim-Skolem property is meaningless over finite structures, a natural finitary analogue of it is satisfied by a broad spectrum of classes of structures of interest and importance in computer science.

In summary, our studies have yielded new notions and results that are of significance over both infinite structures as well as finite structures, and have yielded in both these contexts, a new parameterized generalization of the Łoś-Tarski preservation theorem.