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A generalization of the Łoś–Tarski preservation theorem

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We present new parameterized preservation properties that provide for each natural number k, semantic characterizations of the $\exists^k\forall^*$ and $\forall^k\exists^*$ prefix classes of first order logic sentences, over the class of all structures and for arbitrary finite vocabularies. These properties, that we call *preservation under substructures modulo k-cruxes* and *preservation under k-ary covered extensions* respectively, correspond exactly to the classical properties of preservation under substructures and preservation under extensions, when k equals 0. As a consequence, we get a parameterized generalization of the Łoś–Tarski preservation theorem for sentences, in both its substructural and extensional forms. We call our characterizations collectively the *generalized Łoś–Tarski theorem for sentences*. We generalize this theorem to theories, by showing that theories that are preserved under k-ary covered extensions are characterized by theories of $\forall^k\exists^*$ sentences, and theories that are preserved under substructures modulo k-cruxes, are equivalent, under a well-motivated model-theoretic hypothesis, to theories of $\exists^k\forall^*$ sentences. In contrast to existing preservation properties in the literature that characterize the Σ_2^0 and Π_2^0 prefix classes of FO sentences, our preservation properties are combinatorial and finitary in nature, and stay non-trivial over finite structures as well.

1. Introduction

Preservation theorems in first order logic (henceforth abbreviated FO) have been extensively studied in model theory. An FO preservation theorem for a model-theoretic operation syntactically characterizes elementary classes of structures that are closed under that operation. A classical preservation theorem (also one of the earliest) is the Łoś–Tarski theorem, which states that over the class of all (arbitrary) structures, an FO sentence is preserved under substructures if, and only if, it is equivalent to a universal sentence (see Theorem 3.2.2 in [3]). In dual form, the theorem states that an FO sentence is preserved under extensions if, and only if, it is equivalent to an existential sentence. It is well known that if the vocabulary contains

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only relation symbols, then the sizes of the minimal models of a sentence preserved under extensions are no larger than the number of quantifiers in any equivalent existential sentence. Thus, the dual form of the Łoś–Tarski theorem not only asserts the equivalence of a syntactic and a semantic class of FO sentences, but also yields a relation between a quantitative model-theoretic property (i.e., sizes of minimal models) of a sentence in the semantic class and the count of quantifiers in an equivalent sentence in the syntactic class.

A syntactic subclass of FO that is semantically richer than the universal and existential classes of sentences, is the Σ_2^0 class – the class of all prenex sentences having prefix structure of the form $\exists^*\forall^*$, i.e. sentences whose prefix structure consists of at most two blocks of quantifiers, with the leading block being existential. The literature contains several semantic characterizations, over the class of all structures, for this syntactic class using preservation properties defined in terms of notions such as ascending chains, descending chains, and 1-sandwiches (see Theorem 3.2.3, Proposition 5.2.16 and Theorem 5.2.6 in [3]). These results, in dual form, give semantic characterizations of the Π_2^0 class, which is the class of all $\forall^*\exists^*$ sentences, i.e. prenex sentences whose prefix contains at most two blocks of quantifiers with the leading block being universal. However, none of these characterizations relates quantifier counts in the aforementioned syntactic classes to any model-theoretic properties.

In this paper, we take a step towards addressing this problem. Specifically, we present new preservation theorems that provide semantic characterizations of sentences in prenex normal form, having quantifier prefixes of the form $\exists^k\forall^*$ or $\forall^k\exists^*$, i.e., quantifier prefixes consisting of at most two blocks of quantifiers and in which the leading block has k quantifiers for a given natural number k. Towards these theorems, we introduce, for a given sentence φ and a model $\mathfrak{A}$ of φ, the notions of *a k-crux of* $\mathfrak{A}$ *with respect to* φ and *a substructure of* $\mathfrak{A}$ *modulo a k-crux*. The latter notion corresponds exactly to the classical notion of substructure when k is equal to 0. We define the property of *preservation under substructures modulo k-cruxes* as a natural parameterized generalization of the property of preservation under substructures. Likewise, on the dual front, we introduce the notions of *k-ary covers*, *k-ary covered extensions* and *preservation under k-ary covered extensions*. The latter two notions reduce to the classical notions of extension and preservation under extensions respectively, when k equals 0. Our preservation theorems give syntactic characterizations of the above preservation properties. Specifically, we show for every natural number k, that (i) an FO sentence is preserved under substructures modulo k-cruxes if, and only if, it is equivalent to a prenex sentence having quantifier prefix of the form $\exists^k\forall^*$, and (ii) an FO sentence is preserved under k-ary covered extensions if, and only if, it is equivalent to a prenex sentence having quantifier prefix of the form $\forall^k\exists^*$. To the best of our knowledge, these results, that we collectively call the *generalized Łoś–Tarski theorem for sentences*, are the first to relate natural quantitative properties of models of sentences in a semantic class to counts of leading quantifiers in equivalent $\exists^*\forall^*$ or $\forall^*\exists^*$ sentences. They provide new and finer characterizations of the Σ_2^0 and Π_2^0 prefix classes vis-à-vis the characterizations of these classes in the literature.

In contrast to the existing preservation properties alluded to earlier, that characterize the Σ_2^0 and Π_2^0 classes, our preservation properties are combinatorial and finitary in nature, and stay non-trivial over finite structures as well. There has been a recent renewal of interest in preservation theorems in the context of finite model theory. Since most preservation theorems fail[1] over the class of all finite structures, recent research [1,2,4,6,7] has focused attention on studying classical preservation theorems over 'well-behaved' classes of finite structures. In particular, Atserias, Dawar and Grohe showed in [2] that under suitable closure assumptions, classes of structures that are acyclic or of bounded degree admit the Łoś–Tarski theorem for sentences. They also show the Łoś–Tarski theorem for sentences to be true over the class of all structures of tree-width at most k, for each natural number k (though the theorem is not necessarily true over proper subclasses of these classes). In a recent work [18], we identified many interesting classes of finite structures that admit the generalized Łoś–Tarski theorem for sentences. Specific examples include the classes of words, trees (as partial orders), structures of bounded tree-depth, grids of bounded dimension, various well-known subclasses

[1] A notable exception is the homomorphism preservation theorem [12].

of co-graphs, etc. Some of these, like bounded tree-depth classes (that are proper subclasses of bounded tree-width classes), were earlier not known to even satisfy the Łoś–Tarski theorem. Thus, the preservation properties studied in this paper yield new preservation theorems in the contexts of both classical model theory and finite model theory.

The organization of this paper is as follows.

Section 2: We recall basic notions and the Łoś–Tarski theorem from the literature, and introduce notation used in the paper.

Section 3: We define the properties of 'preservation under substructures modulo k-cruxes' and 'preservation under k-ary covered extensions', denoted $PSC(k)$ and $PCE(k)$ respectively, and formally show their duality.

Section 4: We first provide in subsection 4.1, a syntactic characterization of $PCE(k)$ theories in terms of theories of $\forall^k\exists^*$ sentences. As a corollary, we obtain the generalized Łoś–Tarski theorem for sentences. We next show in subsection 4.2, that $PSC(k)$ theories are equivalent to theories of Σ^0_2 sentences, and that the converse is not true in general, even if all sentences of the Σ^0_2 theory have at most one existential quantifier. We conclude the section by showing in subsection 4.3 that, under a well-motivated model-theoretic hypothesis, $PSC(k)$ theories are equivalent to theories of $\exists^k\forall^*$ sentences (thereby refining the result proved in subsection 4.2). This is done by providing, under the hypothesis, a characterization of $PSC(k)$ theories in terms of sentences of an infinitary logic, and then "compiling" these infinitary sentences "down to" their equivalent FO theories, using suitable *finite approximations* of the former.

Section 5: We define natural generalizations of the $PSC(k)$ and $PCE(k)$ properties, called *preservation under substructures modulo finite cruxes* and *preservation under finitary covered extensions*, respectively denoted PSC_f and PCE_f. We present duality and characterization results for these properties analogous to the results for $PSC(k)$ and $PCE(k)$. We present a comparison of PCE_f and $\bigcup_{k\geq 0} PCE(k)$, and show that these properties surprisingly coincide for sentences, although the former strictly subsumes the latter in the case of theories. For the case of sentences, we show similar results for the relation between PSC_f and $\bigcup_{k\geq 0} PSC(k)$.

Section 6: We present a comparison of our notions and results with related work in the literature.

Section 7: We conclude with discussions and directions for future work.

2. Background

We will be concerned with only FO throughout this paper. We assume that the reader is familiar with standard notation and terminology used in the syntax and semantics of FO (see [3]). A *vocabulary* τ is a set of predicate, function and constant symbols. In this paper, we will always be concerned with arbitrary *finite* vocabularies, unless explicitly stated otherwise. We denote by FO(τ) the set of all FO formulae over vocabulary τ. A sequence $(x_1, \ldots, x_k)$ of variables is denoted by $\bar{x}$. A formula ψ whose free variables are among $\bar{x}$, is denoted by $\psi(\bar{x})$. A formula with no free variables is called a *sentence*. A *theory*, resp. *FO*(τ) *theory*, is a set of sentences, resp. a set of FO(τ) sentences. A theory, resp. FO(τ) theory, *whose free variables are among* $\bar{x}$, is a set of formulae, resp. FO(τ) formulae, all of whose free variables are among $\bar{x}$. We denote by $\mathbb{N}$, the natural numbers *including zero*. We abbreviate a block of quantifiers of the form $Qx_1 \ldots Qx_k$ by $Q^k\bar{x}$ or $Q\bar{x}$ (depending on what is better suited for the context), where $Q \in \{\forall, \exists\}$ and $k \in \mathbb{N}$. By Q^*, we mean a block of k Q quantifiers, for some $k \in \mathbb{N}$. For every non-zero $k \in \mathbb{N}$, we denote by Σ^0_k (resp. Π^0_k), the class of all FO sentences in prenex normal form, whose quantifier prefix begins with $\exists$ (resp. $\forall$) and consists of $k-1$ alternations of quantifiers. We call Σ^0_1 formulae *existential* and Π^0_1 formulae *universal*. We call Σ^0_2 formulae with k existential quantifiers $\exists^k\forall^*$ *formulae*, and Π^0_2 formulae with k universal quantifiers $\forall^k\exists^*$ *formulae*. We use the standard notions of τ-structures (denoted $\mathfrak{A}, \mathfrak{B}$ etc.; we refer to these simply as structures when τ is clear from context), substructures (denoted as $\mathfrak{A} \subseteq \mathfrak{B}$), extensions, isomorphisms (denoted $\mathfrak{A} \cong \mathfrak{B}$), elementary equivalence (denoted $\mathfrak{A} \equiv \mathfrak{B}$), elementary substructures (denoted $\mathfrak{A} \preceq \mathfrak{B}$)

and elementary extensions, as defined in [3], and study preservation theorems over the class of *all (arbitrary) structures*. By the *size* (or *power*) of a structure $\mathfrak{A}$, we mean the cardinality of its universe, and denote it by $|\mathfrak{A}|$.

We first recall the classical dual notions of preservation under substructures and preservation under extensions. We fix a vocabulary τ in our discussion below.

Definition 2.1. Let $\mathcal{U}$ be a class of structures.

1. A subclass $\mathcal{S}$ of $\mathcal{U}$ is said to be *preserved under substructures over* $\mathcal{U}$, abbreviated as $\mathcal{S}$ *is PS over* $\mathcal{U}$, if for each structure $\mathfrak{A} \in \mathcal{S}$, if $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B} \in \mathcal{U}$, then $\mathfrak{B} \in \mathcal{S}$.
2. A subclass $\mathcal{S}$ of $\mathcal{U}$ is said to be *preserved under extensions over* $\mathcal{U}$, abbreviated as $\mathcal{S}$ *is PE over* $\mathcal{U}$, if for each structure $\mathfrak{A} \in \mathcal{S}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B} \in \mathcal{U}$, then $\mathfrak{B} \in \mathcal{S}$.

If V and T are theories, then we say T *is PS modulo* V (resp. T *is PE modulo* V) if the class of models of $T \cup V$ is *PS* (resp. *PE*) over the class of models of V. For a sentence ϕ, we say ϕ *is PS modulo* V (resp. ϕ *is PE modulo* V) if the theory $\{\phi\}$ is *PS* (resp. *PE*) modulo V.

As an example, let $\tau = \{E\}$ be the vocabulary consisting of a single relation symbol E that is binary, and let $\mathcal{U}$ be the class of all τ-structures in which E is interpreted as a symmetric binary relation. The class $\mathcal{U}$ can be seen as the class of all undirected graphs. Let $\mathcal{S}_1$ be the subclass of $\mathcal{U}$ consisting of all undirected graphs that are acyclic. Let $\mathcal{S}_2$ be the subclass of $\mathcal{U}$ consisting of all undirected graphs that contain a triangle as a subgraph. It is easy to see that $\mathcal{S}_1$ is *PS* over $\mathcal{U}$, and $\mathcal{S}_2$ is *PE* over $\mathcal{U}$. Observe that $\mathcal{U}$ is defined by the theory $V = \{\forall x \forall y\, (E(x, y) \rightarrow E(y, x))\}$. Let ψ_n be the universal sentence that asserts the absence of a cycle of length n as a subgraph. Then $\mathcal{S}_1$ is exactly the class of models in $\mathcal{U}$, of the theory $T = \{\psi_n \mid n \geq 3\}$, and $\mathcal{S}_2$ is exactly the class of models in $\mathcal{U}$, of the sentence $\phi = \neg\psi_3$. Whereby, T is *PS* modulo V, and ϕ is *PE* modulo V.

The following lemma shows the duality between *PS* and *PE*. Below, ‘iff’ denotes ‘if and only if’.

Lemma 2.2 *(PS–PE duality). Let* $\mathcal{U}$ *be a class of structures,* $\mathcal{S}$ *be a subclass of* $\mathcal{U}$ *and* $\overline{\mathcal{S}}$ *be the complement of* $\mathcal{S}$ *in* $\mathcal{U}$*. Then* $\mathcal{S}$ *is PS over* $\mathcal{U}$ *iff* $\overline{\mathcal{S}}$ *is PE over* $\mathcal{U}$*. In particular, if* $\mathcal{U}$ *is defined by a theory* V*, then a sentence* ϕ *is PS modulo* V *iff* $\neg\phi$ *is PE modulo* V*.*

The notion of a theory being *PS* modulo V or *PE* modulo V can be extended to theories with free variables in a natural manner. Given $n \in \mathbb{N}$, denote by τ_n, the vocabulary obtained by expanding τ with n fresh and distinct constants symbols $c_1, \ldots, c_n$. Let $T(\bar{x})$ be an FO(τ) theory with free variables among $\bar{x} = (x_1, \ldots, x_n)$, and let T' be the FO(τ_n) theory obtained by substituting c_i for x_i in $T(\bar{x})$, for each $i \in \{1, \ldots, n\}$. Given a theory V, we say $T(\bar{x})$ *is PS modulo* V if T' is *PS* modulo V, where V is treated as an FO(τ_n) theory. The notion $T(\bar{x})$ *is PE modulo* V is defined similarly.

In the late ’40s, Jerzy Łoś and Alfred Tarski provided syntactic characterizations of theories that are *PS* and theories that are *PE* via the following preservation theorem. This result and its proof set the trend for various other preservation theorems to follow.

Theorem 2.3 *(Łoś–Tarski, 1949–1950). Let* $T(\bar{x})$ *be a theory whose free variables are among* $\bar{x}$*. Given a theory* V*, each of the following is true.*

1. $T(\bar{x})$ *is PS modulo* V *iff* $T(\bar{x})$ *is equivalent modulo* V *to a theory* $Y(\bar{x})$ *of universal formulae, all of whose free variables are among* $\bar{x}$*. If* $T(\bar{x})$ *is a singleton, then so is* $Y(\bar{x})$*.*
2. $T(\bar{x})$ *is PE modulo* V *iff* $T(\bar{x})$ *is equivalent modulo* V *to a theory* $Y(\bar{x})$ *of existential formulae, all of whose free variables are among* $\bar{x}$*. If* $T(\bar{x})$ *is a singleton, then so is* $Y(\bar{x})$*.*

In the remainder of the paper, if $\mathcal{U}$, as mentioned in the definitions above, is clear from context, then we skip mentioning its associated qualifier, namely, 'over $\mathcal{U}$'. Likewise, we skip mentioning 'modulo V' when V is clear from context.

3. Parameterized preservation properties generalizing *PS* and *PE*

3.1. Preservation under substructures modulo k-cruxes

Definition 3.1. Let $\mathcal{U}$ be a class of structures and $k \in \mathbb{N}$. A subclass $\mathcal{S}$ of $\mathcal{U}$ is said to be *preserved under substructures modulo k-cruxes over* $\mathcal{U}$, abbreviated as $\mathcal{S}$ *is* $PSC(k)$ *over* $\mathcal{U}$, if for every structure $\mathfrak{A} \in \mathcal{S}$, there exists a subset C of the universe of $\mathfrak{A}$, of size at most k, such that if $\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B}$ contains C and $\mathfrak{B} \in \mathcal{U}$, then $\mathfrak{B} \in \mathcal{S}$. The set C is called a *k-crux of* $\mathfrak{A}$ *with respect to* $\mathcal{S}$ *over* $\mathcal{U}$. Any substructure $\mathfrak{B}$ of $\mathfrak{A}$, that contains C is called a *substructure of* $\mathfrak{A}$ *modulo the k-crux* C. Given theories V and T, we say T *is* $PSC(k)$ *modulo* V, if the class of models of $T \cup V$ is $PSC(k)$ over the class of models of V. For a sentence ϕ, we say ϕ *is* $PSC(k)$ *modulo* V if the theory $\{\phi\}$ is $PSC(k)$ modulo V.

Let $\mathcal{S}$, $\mathcal{U}$, $\mathfrak{A}$, C, V, T and ϕ be as above. We abbreviate 'with respect to' as 'w.r.t.' henceforth. If $\mathcal{U}$ is defined by V and $\mathcal{S}$ is defined by T over $\mathcal{U}$, then we say C is a *k-crux of* $\mathfrak{A}$ *w.r.t.* T *modulo* V. If $\mathcal{S}$ is defined by ϕ over $\mathcal{U}$, then we say C is a *k-crux of* $\mathfrak{A}$ *w.r.t.* ϕ *modulo* V. As in the previous section, if any of $\mathcal{S}$, $\mathcal{U}$, T, V or ϕ is clear from context, then we skip mentioning its associated qualifier (viz., 'w.r.t. $\mathcal{S}$', 'over $\mathcal{U}$', 'w.r.t. T', 'modulo V' and 'w.r.t. ϕ' respectively) in the definitions above.

Remark 3.2. Note that Definition 3.1 is an adapted version of related definitions in [15,17] and [14]. The notion of 'core' in Definition 1 of [17] is exactly the notion of 'crux' defined above, when the underlying class $\mathcal{U}$ is the class of all structures. We avoid using the word 'core' for a crux to prevent confusion with existing notions of cores in the literature [1,8,12].

Example 3.3. Let the underlying class $\mathcal{U}$ be the class of all undirected graphs. Given $k \in \mathbb{N}$, consider the class $\mathcal{S}_k$ of all graphs of $\mathcal{U}$ containing a cycle of length k as a subgraph. Clearly, for any graph G in $\mathcal{S}_k$, the vertices of any cycle of length k in G form a k-crux of G w.r.t. $\mathcal{S}_k$. Hence $\mathcal{S}_k$ is $PSC(k)$. It is easy to see that $\mathcal{S}_k$ is definable by an FO sentence, call it ϕ, whereby ϕ is $PSC(k)$.

Fix an underlying class $\mathcal{U}$ of structures. For properties P_1 and P_2 of subclasses of $\mathcal{U}$, we denote by $P_1 \Rightarrow P_2$ that any subclass of $\mathcal{U}$ satisfying P_1 also satisfies P_2. We denote by $P_1 \Leftrightarrow P_2$ that $P_1 \Rightarrow P_2$ and $P_2 \Rightarrow P_1$. It is now easy to check the following facts concerning the $PSC(k)$ subclasses of $\mathcal{U}$: (i) $PSC(0)$ coincides with the property of preservation under substructures, so $PSC(0) \Leftrightarrow PS$, (ii) $PSC(l) \Rightarrow PSC(k)$ for $l \leq k$. If $\mathcal{U}$ is any substructure-closed class that contains infinitely many finite structures, then for each l, there exists $k > l$ and a $PSC(k)$ subclass $\mathcal{S}$ of $\mathcal{U}$ such that $\mathcal{S}$ is not $PSC(l)$ over $\mathcal{U}$. This is seen as follows. Given l, let $k > l$ be such that there is some structure of size k in $\mathcal{U}$, and let ϕ_k be the sentence asserting that there are at least k elements in any model. Clearly ϕ_k is $PSC(k)$ over $\mathcal{U}$ but not $PSC(l)$ over $\mathcal{U}$.

Define the property PSC of subclasses of $\mathcal{U}$ as follows: A subclass $\mathcal{S}$ of $\mathcal{U}$ is PSC *over* $\mathcal{U}$ if it is $PSC(k)$ over $\mathcal{U}$ for some $k \in \mathbb{N}$. Notationally, $PSC \Leftrightarrow \bigcup_{k \geq 0} PSC(k)$. If $\mathcal{U}$ is defined by a theory V, then the notions of 'a sentence is PSC modulo V' and 'a theory is PSC modulo V' are defined similarly as in Definition 3.1. The implications mentioned in the previous paragraph show that PSC generalizes PS. If $\mathcal{U}$ is any substructure-closed class that contains infinitely many finite structures, then the strict implications mentioned above show a strictly infinite hierarchy within PSC; whence the latter provides a strict generalization of PS.

Suppose that $\mathcal{U}$ is defined by a theory V. Given a Σ^0_2 sentence $\phi = \exists x_1 \ldots \exists x_k \, \forall \bar{y} \, \xi(x_1, \ldots, x_k, \bar{y})$ and a structure $\mathfrak{A}$ of $\mathcal{U}$ such that $\mathfrak{A} \models \phi$, any set of *witnesses* in $\mathfrak{A}$ of the existential quantifiers of ϕ, forms a k-crux of $\mathfrak{A}$. In particular, if $a_1, \ldots, a_k$ are witnesses in $\mathfrak{A}$, of the quantifiers associated with $x_1, \ldots, x_k$ (whence $\mathfrak{A} \models \forall \bar{y}\ \xi(a_1, a_2, \ldots, a_k, \bar{y})$), then given any substructure $\mathfrak{B}$ of $\mathfrak{A}$ containing $a_1, \ldots, a_k$, the latter elements can again be chosen as witnesses in $\mathfrak{B}$, to make ϕ true in $\mathfrak{B}$. Therefore, ϕ is $PSC(k)$ (modulo V).

Remark 3.4. Contrary to intuition, witnesses and k-cruxes cannot always be equated! Consider the sentence $\phi = \exists x \forall y E(x, y)$ and the structure $\mathfrak{A} = (\mathbb{N}, \leq)$, i.e. the natural numbers with the usual ordering. Let $\mathcal{U}$ be the class of all structures. Clearly, ϕ is $PSC(1)$, $\mathfrak{A} \models \phi$ and the only witness of the existential quantifier of ϕ in $\mathfrak{A}$ is the minimum element $0 \in \mathbb{N}$. In contrast, every singleton subset of $\mathbb{N}$ is a 1-crux of $\mathfrak{A}$ because each substructure of $\mathfrak{A}$ contains a minimum element under the induced order; this in turn is due to $\leq$ being a well-ordering of $\mathbb{N}$. This example shows that there can be models in which there are many more (even infinitely more) cruxes than witnesses.

Since Σ^0_1 and Π^0_1 sentences are also Σ^0_2 sentences and the latter are PSC, the former are also PSC. However, Π^0_2 sentences are not necessarily PSC. Consider $\phi = \forall x \exists y E(x, y)$ and consider the model $\mathfrak{A}$ of ϕ given by $\mathfrak{A} = (\mathbb{N}, E^{\mathfrak{A}} = \{(i, i+1) \mid i \in \mathbb{N}\})$. It is easy to check that no finite substructure of $\mathfrak{A}$ models ϕ; then $\mathfrak{A}$ does not have any k-crux for any $k \in \mathbb{N}$, whence ϕ is not $PSC(k)$ for any k, and hence is not PSC.

3.2. Preservation under k-ary covered extensions

The classical notion of "extension of a structure" has a natural generalization to the notion of *extension of a collection of structures* as follows. A structure $\mathfrak{A}$ is said to be an extension of a collection R of structures if for each $\mathfrak{B} \in R$, we have $\mathfrak{B} \subseteq \mathfrak{A}$. We now define a special kind of extensions of a collection of structures.

Definition 3.5. For $k \in \mathbb{N}$, a structure $\mathfrak{A}$ is said to be a *k-ary covered extension* of a non-empty collection R of structures if (i) $\mathfrak{A}$ is an extension of R, and (ii) for every subset C of the universe of $\mathfrak{A}$, of size at most k, there is a structure in R that contains C. We call R a *k-ary cover* of $\mathfrak{A}$.

Example 3.6. Let $\mathfrak{A}$ be a graph on n vertices and let R be the collection of all r sized induced subgraphs of $\mathfrak{A}$, where $1 \leq r \leq n$. Then $\mathfrak{A}$ is a k-ary covered extension of R for every k in $\{0, \ldots, r\}$.

Remark 3.7. Note that a 0-ary covered extension of R is simply an extension of R. For $k > 0$, the universe of a k-ary covered extension of R is necessarily the union of the universes of the structures in R. However, different k-ary covered extensions of R can differ in the interpretation of predicates (if any) of arity greater than k. Note also that a k-ary covered extension of R is an l-ary covered extension of R for every $l \in \{0, \ldots, k\}$.

Definition 3.8. Let $\mathcal{U}$ be a class of structures and $k \in \mathbb{N}$. A subclass $\mathcal{S}$ of $\mathcal{U}$ is said to be *preserved under k-ary covered extensions* over $\mathcal{U}$, abbreviated as $\mathcal{S}$ *is* $PCE(k)$ *over* $\mathcal{U}$, if for every collection R of structures of $\mathcal{S}$, if $\mathfrak{A}$ is a k-ary covered extension of R and $\mathfrak{A} \in \mathcal{U}$, then $\mathfrak{A} \in \mathcal{S}$. Given theories V and T, we say T *is* $PCE(k)$ *modulo* V if the class of models of $T \cup V$ is $PCE(k)$ over the class of models of V. For a sentence ϕ, we say ϕ *is* $PCE(k)$ *modulo* V if the theory $\{\phi\}$ is $PCE(k)$ modulo V.

As in the previous subsection, if any of $\mathcal{U}$ or V is clear from context, then we skip mentioning its associated qualifier. The following lemma establishes the duality between $PSC(k)$ and $PCE(k)$, generalizing the duality between PS and PE given by Lemma 2.2.

Lemma 3.9 *($PSC(k)$–$PCE(k)$ duality). Let $\mathcal{U}$ be a class of structures, $\mathcal{S}$ be a subclass of $\mathcal{U}$ and $\overline{\mathcal{S}}$ be the complement of $\mathcal{S}$ in $\mathcal{U}$. Then $\mathcal{S}$ is $PSC(k)$ over $\mathcal{U}$ iff $\overline{\mathcal{S}}$ is $PCE(k)$ over $\mathcal{U}$, for each $k \in \mathbb{N}$. In particular, if $\mathcal{U}$ is defined by a theory V, then a sentence ϕ is $PSC(k)$ modulo V iff $\neg\phi$ is $PCE(k)$ modulo V.*

Proof. If: Suppose $\overline{\mathcal{S}}$ is $PCE(k)$ over $\mathcal{U}$ but $\mathcal{S}$ is not $PSC(k)$ over $\mathcal{U}$. Then there exists $\mathfrak{A} \in \mathcal{S}$ such that for every set C of at most k elements of $\mathfrak{A}$, there is a substructure $\mathfrak{B}_C$ of $\mathfrak{A}$ that (i) contains C, and (ii) belongs to $\mathcal{U} \setminus \mathcal{S}$, i.e. belongs to $\overline{\mathcal{S}}$. Then $R = \{\mathfrak{B}_C \mid C \text{ is a subset of } \mathfrak{A}, \text{ of size at most } k\}$ is a k-ary cover of $\mathfrak{A}$. Since $\overline{\mathcal{S}}$ is $PCE(k)$ over $\mathcal{U}$, it follows that $\mathfrak{A} \in \overline{\mathcal{S}}$ – a contradiction.

Only If: Suppose $\mathcal{S}$ is $PSC(k)$ over $\mathcal{U}$ but $\overline{\mathcal{S}}$ is not $PCE(k)$ over $\mathcal{U}$. Then there exists $\mathfrak{A} \in \mathcal{S}$ and a k-ary cover R of $\mathfrak{A}$ such that every structure $\mathfrak{B}$ of R belongs to $\overline{\mathcal{S}}$. Since $\mathcal{S}$ is $PSC(k)$ over $\mathcal{U}$, there exists a k-crux C of $\mathfrak{A}$ w.r.t. $\mathcal{S}$ over $\mathcal{U}$. Consider the structure $\mathfrak{B}_C \in R$ that contains C – this exists since R is a k-ary cover of $\mathfrak{A}$. Then $\mathfrak{B}_C \in \mathcal{S}$ since C is a k-crux of $\mathfrak{A}$ – a contradiction. □

Fix an underlying class $\mathcal{U}$ of structures. Analogous to the notion of PSC, we define the notion of PCE as $PCE \Leftrightarrow \bigcup_{k \geq 0} PCE(k)$. The notion of a class, a sentence and a theory, being PCE is defined analogously to corresponding notions for PSC. Then from the observations in the previous subsection, Remark 3.7 and Lemma 3.9 above, we see that (i) $PCE(0) \Leftrightarrow PE$, (ii) $PCE(l) \Rightarrow PCE(k)$ for $l \leq k$, and (iii) a subclass $\mathcal{S}$ of $\mathcal{U}$ is PSC over $\mathcal{U}$ iff the complement $\overline{\mathcal{S}}$ of $\mathcal{S}$ in $\mathcal{U}$, is PCE over $\mathcal{U}$. Further, if $\mathcal{U}$ is defined by a theory V, then all Π^0_2 sentences having at most k universal quantifiers are $PCE(k)$ (modulo V) and hence PCE, whereby all Σ^0_1 and Π^0_1 sentences are PCE as well. However Σ^0_2 sentences, in general, are not PCE since, as seen towards the end of the previous subsection, Π^0_2 sentences are, in general, not PSC.

4. Syntactic characterizations

Given an FO(τ) theory $T(\bar{x})$ whose free variables are among $\bar{x} = (x_1, \ldots, x_n)$, we first define the notion of $T(\bar{x})$ being $\mathcal{P}$ modulo V, for a given theory V, where $\mathcal{P} \in \{PSC(k), PSC, PCE(k), PCE\}$. As in Section 2, let $c_1, \ldots, c_n$ be the distinct constant symbols of $\tau_n \setminus \tau$, and let T' be the FO(τ_n) theory obtained by substituting c_i for x_i in $T(\bar{x})$, for each $i \in \{1, \ldots, n\}$. Then we say *$T(\bar{x})$ is $\mathcal{P}$ modulo V* if T' is $\mathcal{P}$ modulo V, where V is treated as an FO(τ_n) theory.

4.1. Characterization of $PCE(k)$ theories

The central result of this subsection is as follows.

Theorem 4.1. *Let V and $T(\bar{x})$ be theories, and $k \in \mathbb{N}$. Then $T(\bar{x})$ is $PCE(k)$ modulo V iff $T(\bar{x})$ is equivalent modulo V to a theory of Π^0_2 formulae, all of whose free variables are among $\bar{x}$, and all of which have k universal quantifiers.*

We prove Theorem 4.1 for theories T without any free variables; the proof for theories with free variables follows from definitions. Our proof below crucially involves the use of *λ-saturated structures*, so we refer the reader to Chapter 5 of [3] for all the results concerning them that we make use of.

We first introduce some terminology and notation. Given theories T and V, we say that Γ is the *set of $\forall^k\exists^*$ consequences of T modulo V* if $\Gamma = \{\varphi \mid \varphi \text{ is a } \forall^k\exists^* \text{ sentence and } (V \cup T) \vdash \varphi\}$. Let $\mathfrak{A}$ be a structure and $\bar{a}$ be a k-tuple of elements of $\mathfrak{A}$. We denote by $\mathsf{th}(\mathfrak{A})$, the theory of $\mathfrak{A}$, i.e. the set of all FO sentences that are true in $\mathfrak{A}$. We let $\mathsf{tp}_{\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$ denote the FO-type of $\bar{a}$ in $\mathfrak{A}$, i.e. the set of all FO formulae, all of whose free variables are among $x_1, \ldots, x_k$, that are true of $\bar{a}$ in $\mathfrak{A}$. By $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$, we denote the Π^0_1-type of $\bar{a}$ in $\mathfrak{A}$, i.e. the subset of $\mathsf{tp}_{\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$ that consists of all Π^0_1 formulae of $\mathsf{tp}_{\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$. The following lemma is key to the proof.

Lemma 4.2. *Let V and T be consistent theories, and $k \in \mathbb{N}$. Let Γ be the set of $\forall^k\exists^*$ consequences of T modulo V. Then for all infinite cardinals λ, for every λ-saturated structure $\mathfrak{A}$ that models V, we have that $\mathfrak{A} \models \Gamma$ iff there exists a k-ary cover R of $\mathfrak{A}$ such that $\mathfrak{B} \models (V \cup T)$ for every $\mathfrak{B} \in R$.*

Proof. The 'If' direction is easy: for each $\mathfrak{B} \in R$, since $\mathfrak{B} \models (V \cup T)$, we have $\mathfrak{B} \models \varphi$ for each $\varphi \in \Gamma$. From the discussion in Section 3.2, any $\forall^k\exists^*$ sentence is $PCE(k)$ modulo V. Then since R is a k-ary cover of $\mathfrak{A}$, we have $\mathfrak{A} \models \varphi$ for each $\varphi \in \Gamma$.

For the 'Only If' direction, let the vocabulary of V and T be τ. We show that for every k-tuple $\bar{a}$ of $\mathfrak{A}$, there is a substructure $\mathfrak{A}_{\bar{a}}$ of $\mathfrak{A}$ containing (the elements of) $\bar{a}$ such that $\mathfrak{A}_{\bar{a}} \models (V \cup T)$. Then the set $R = \{\mathfrak{A}_{\bar{a}} \mid \bar{a} \text{ is a } k\text{-tuple of } \mathfrak{A}\}$ forms the desired k-ary cover of $\mathfrak{A}$. To show the existence of $\mathfrak{A}_{\bar{a}}$, it suffices to show that there exists a τ-structure $\mathfrak{B}$ such that (i) $|\mathfrak{B}| \leq \lambda$, (ii) $\mathfrak{B} \models (V \cup T)$, and (iii) the Π^0_1-type of $\bar{a}$ in $\mathfrak{A}$, namely $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$, is realized in $\mathfrak{B}$ by some k-tuple, say $\bar{b}$. Then every Σ^0_1 sentence of $\mathrm{FO}(\tau_k)$ true in $(\mathfrak{B}, \bar{b})$ is also true in $(\mathfrak{A}, \bar{a})$. Since $\mathfrak{A}$ is λ-saturated, we have that $(\mathfrak{A}, \bar{a})$ is also λ-saturated. There exists then, an isomorphic embedding $f : (\mathfrak{B}, \bar{b}) \to (\mathfrak{A}, \bar{a})$, whereby the τ-reduct of the image of $(\mathfrak{B}, \bar{b})$ under f can serve as $\mathfrak{A}_{\bar{a}}$. The proof is therefore completed by showing the existence of $\mathfrak{B}$ with the above properties.

Suppose $Z(x_1, \ldots, x_k) = V \cup T \cup \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$ is inconsistent. By the compactness theorem, there is a finite subset of $Z(x_1, \ldots, x_k)$ that is inconsistent. Since $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$ is closed under taking finite conjunctions and since each of $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$, V and T is consistent, there is a formula $\psi(x_1, \ldots, x_k)$ in $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$ such that $V \cup T \cup \{\psi(x_1, \ldots, x_k)\}$ is inconsistent. In other words, $(V \cup T) \vdash \neg\psi(x_1, \ldots, x_k)$. By $\forall$-introduction, we have $(V \cup T) \vdash \varphi$, where $\varphi = \forall x_1 \ldots \forall x_k \neg\psi(x_1, \ldots, x_k)$. Observe that φ is a $\forall^k\exists^*$ sentence; then by the definition of Γ, we have $\varphi \in \Gamma$, and hence $\mathfrak{A} \models \varphi$. Instantiating the k-tuple $(x_1, \ldots, x_k)$ as $\bar{a}$, we have $(\mathfrak{A}, \bar{a}) \models \neg\psi(x_1, \ldots, x_k)$, contradicting the fact that $\psi(x_1, \ldots, x_k) \in \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1, \ldots, x_k)$. Then $Z(x_1, \ldots, x_k)$ must be consistent. By the downward Löwenheim–Skolem theorem, there is a model $(\mathfrak{B}, \bar{b})$ of $Z(x_1, \ldots, x_k)$ of power at most λ; then $\mathfrak{B}$ is as desired. □

The following lemma is straightforward.

Lemma 4.3. *Let $\mathcal{U}$ be a class of structures. For an index set I, let $\{\mathcal{S}_i \mid i \in I\}$ be a collection of subclasses of $\mathcal{U}$ such that $\mathcal{S}_i$ is $PCE(k)$ over $\mathcal{U}$, for each $i \in I$. Then $\bigcap_{i \in I} \mathcal{S}_i$ is $PCE(k)$ over $\mathcal{U}$.*

We can now prove Theorem 4.1 as follows.

Proof of Theorem 4.1. Suppose T is equivalent modulo V to a theory $Y = \{\varphi_i \mid \varphi_i \text{ is a } \forall^k\exists^* \text{ sentence}, i \geq 1\}$. Then φ_i is $PCE(k)$ modulo V for each $i \geq 1$. It follows from Lemma 4.3 that Y is $PCE(k)$ modulo V, whereby T is $PCE(k)$ modulo V.

In the converse direction, suppose T is $PCE(k)$ modulo V. If $V \cup T$ is unsatisfiable, we are trivially done. Otherwise, let Γ be the set of $\forall^k\exists^*$ consequences of T modulo V. Then $(V \cup T) \vdash \Gamma$. We show below that $(V \cup \Gamma) \vdash T$, thereby showing that T is equivalent to Γ modulo V. Suppose $\mathfrak{A} \models (V \cup \Gamma)$. Consider a λ-saturated elementary extension $\mathfrak{A}^+$ of $\mathfrak{A}$, for some $\lambda \geq \omega$. Then $\mathfrak{A}^+ \models (V \cup \Gamma)$. By Lemma 4.2, there exists a k-ary cover R of $\mathfrak{A}^+$ such that $\mathfrak{B} \models (V \cup T)$ for every $\mathfrak{B} \in R$. Since T is $PCE(k)$ modulo V, it follows that $\mathfrak{A}^+ \models T$. Finally, since $\mathfrak{A} \preceq \mathfrak{A}^+$, we have $\mathfrak{A} \models T$. □

An important corollary of Theorem 4.1 is as stated below.

Corollary 4.4. *Given a theory V, a formula $\phi(\bar{x})$ is $PCE(k)$ modulo V iff $\phi(\bar{x})$ is equivalent modulo V to a Π^0_2 formula whose free variables are among $\bar{x}$, and that has k universal quantifiers.*

Proof. Follows from the compactness theorem and the fact that a finite conjunction of $\forall^k\exists^*$ formulae is equivalent to a single $\forall^k\exists^*$ formula. □

4.2. Characterization of $PSC(k)$ theories

The central result of this subsection is as follows.

Theorem 4.5. *Let V and $T(\bar{x})$ be theories, and suppose that $T(\bar{x})$ is $PSC(k)$ modulo V for some $k \in \mathbb{N}$. Then $T(\bar{x})$ is equivalent modulo V to a theory of Σ_2^0 formulae, all of whose free variables are among $\bar{x}$.*

The converse of the above theorem is however not true. Lemma 4.12 presented towards the end of this subsection, gives an example of a theory of Σ_2^0 sentences, each of whose sentences contains only one existential quantifier, and that is not $PSC(k)$ for any $k \in \mathbb{N}$. For the case of sentences however, we have the following characterization, that follows directly from Corollary 4.4 and Lemma 3.9.

Corollary 4.6. *Given a theory V, a formula $\phi(\bar{x})$ is $PSC(k)$ modulo V iff $\phi(\bar{x})$ is equivalent modulo V to a Σ_2^0 formula whose free variables are among $\bar{x}$, and that has k existential quantifiers.*

The approach of 'dualizing' adopted in proving Corollary 4.6 cannot work for characterizing theories that are $PSC(k)$ since the negation of an FO theory might, in general, not be equivalent to any FO theory. Although it is unclear at present what syntactic fragment of FO theories serves as a characterization for $PSC(k)$ theories, Theorem 4.5 shows that such a syntactic fragment is semantically contained inside Σ_2^0 theories.

We prove Theorem 4.5 for theories without free variables. The proof for theories with free variables follows from definitions. Towards the proof, we recall the notion of *sandwiches* as defined by Keisler in [9]. We say that a triple $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ of structures is a *sandwich* if $\mathfrak{A} \preceq \mathfrak{C}$ and $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$. Given structures $\mathfrak{A}$ and $\mathfrak{B}$, we say that $\mathfrak{B}$ *is sandwiched by* $\mathfrak{A}$ if there exist structures $\mathfrak{A}'$ and $\mathfrak{B}'$ such that (i) $\mathfrak{B} \preceq \mathfrak{B}'$ and (ii) $(\mathfrak{A}, \mathfrak{B}', \mathfrak{A}')$ is a sandwich. Given theories V and T, we say T is *preserved under sandwiches by models of T modulo V* if for each model $\mathfrak{A}$ of $V \cup T$, if $\mathfrak{B}$ is sandwiched by $\mathfrak{A}$ and $\mathfrak{B}$ models V, then $\mathfrak{B}$ models T. The following theorem of Keisler (Corollary 5.2 of [9]) gives a syntactic characterization of the aforesaid preservation property in terms of Σ_2^0 theories.

Theorem 4.7 *(Keisler, 1960). Let V and T be theories. Then T is preserved under sandwiches by models of T modulo V iff T is equivalent modulo V to a theory of Σ_2^0 sentences.*

To prove Theorem 4.5, it then suffices to show that if T is $PSC(k)$ modulo V, then T is preserved under sandwiches by models of T modulo V. To do this, we first prove the following lemmas.

Lemma 4.8 *(Sandwich by saturated structures). Let $\mathfrak{A}_1$ and $\mathfrak{B}_1$ be structures such that $\mathfrak{B}_1$ is sandwiched by $\mathfrak{A}_1$. Then for each $\lambda \geq \omega$, for every λ-saturated elementary extension $\mathfrak{A}$ of $\mathfrak{A}_1$, there exists a structure $\mathfrak{B}$ isomorphic to $\mathfrak{B}_1$ such that $\mathfrak{B}$ is sandwiched by $\mathfrak{A}$.*

Lemma 4.9 *(Preservation under sandwich by saturated models). Let V and T be theories such that T is $PSC(k)$ modulo V for some $k \in \mathbb{N}$. Let $\mathfrak{A}$ be a λ-saturated model of $V \cup T$, for some $\lambda \geq \omega$, and let $\mathfrak{B}$ be a model of V. If $\mathfrak{B}$ is sandwiched by $\mathfrak{A}$, then $\mathfrak{B}$ is a model of T.*

Using the above lemmas, we can prove Theorem 4.5, as follows.

Proof of Theorem 4.5. Let T be $PSC(k)$ modulo V. It suffices to show that T is preserved under sandwiches by models of T modulo V. Suppose $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are given structures such that $\mathfrak{B}_1$ is sandwiched by $\mathfrak{A}_1$, $\mathfrak{B}_1 \models V$ and $\mathfrak{A}_1 \models V \cup T$. Consider a λ-saturated elementary extension $\mathfrak{A}$ of $\mathfrak{A}_1$, for some $\lambda \geq \omega$. By Lemma 4.8, there exists a structure $\mathfrak{B}$ isomorphic to $\mathfrak{B}_1$ such that $\mathfrak{B}$ is sandwiched by $\mathfrak{A}$. Then $\mathfrak{A} \models (V \cup T)$

and $\mathfrak{B} \models V$, whence by Lemma 4.9, we have $\mathfrak{B} \models T$. Since $\mathfrak{B}_1 \cong \mathfrak{B}$, we have $\mathfrak{B}_1 \models T$, completing the proof. □

We now prove Lemmas 4.8 and 4.9. We first introduce some notation. Given a τ-structure $\mathfrak{A}$, we denote by $\tau_{\mathfrak{A}}$ the vocabulary obtained by expanding τ with $|\mathfrak{A}|$ fresh constants – one constant per element of $\mathfrak{A}$. Given a τ-structure $\mathfrak{B}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$, we denote by $\mathfrak{B}_{\mathfrak{A}}$, the $\tau_{\mathfrak{A}}$-structure whose τ-reduct is $\mathfrak{B}$ and in which the constant in $\tau_{\mathfrak{A}} \setminus \tau$ corresponding to an element a of $\mathfrak{A}$, is interpreted as a itself. In particular therefore, $\mathfrak{A}_{\mathfrak{A}}$ is a $\tau_{\mathfrak{A}}$-structure whose τ-reduct is $\mathfrak{A}$ and in which every element of the universe is an interpretation of exactly one constant in $\tau_{\mathfrak{A}} \setminus \tau$. By Diag($\mathfrak{A}$), resp. El-diag($\mathfrak{A}$), we mean the diagram, resp. elementary diagram of $\mathfrak{A}$, i.e. the set of all quantifier-free FO($\tau_{\mathfrak{A}}$) sentences, resp. arbitrary FO($\tau_{\mathfrak{A}}$) sentences, that are true in $\mathfrak{A}_{\mathfrak{A}}$. Observe that each of Diag($\mathfrak{A}$) and El-diag($\mathfrak{A}$) is closed under finite conjunctions. Finally, $\mathfrak{A} \preceq_1 \mathfrak{B}$ denotes that (i) $\mathfrak{A} \subseteq \mathfrak{B}$ and (ii) every Σ^0_1 sentence of FO($\tau_{\mathfrak{A}}$) true in $\mathfrak{B}_{\mathfrak{A}}$ is also true in $\mathfrak{A}_{\mathfrak{A}}$.

Lemma 4.10. $\mathfrak{A} \preceq_1 \mathfrak{B}$ *iff there exists* $\mathfrak{A}'$ *such that* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{A}')$ *is a sandwich.*

Proof. The 'If' direction follows easily from the definition of elementary substructure and the fact that existential formulae are preserved under extensions. For the converse, suppose that $\mathfrak{A} \preceq_1 \mathfrak{B}$. Let the vocabularies $\tau_{\mathfrak{B}}$ and $\tau_{\mathfrak{A}}$ be such that for every element a of $\mathfrak{A}$, the constant in $\tau_{\mathfrak{B}}$ corresponding to a is *the same* as the constant in $\tau_{\mathfrak{A}}$ corresponding to a (and hence the constants in $\tau_{\mathfrak{B}} \setminus \tau_{\mathfrak{A}}$ correspond exactly to the elements in $\mathfrak{B}$ that are not in $\mathfrak{A}$). Now consider the theory Y given by $Y = \text{Diag}(\mathfrak{B}) \cup \text{El-diag}(\mathfrak{A})$. Any non-empty finite subset of Diag($\mathfrak{B}$), resp. El-diag($\mathfrak{A}$), is satisfied in $\mathfrak{B}_{\mathfrak{B}}$, resp. $\mathfrak{A}_{\mathfrak{A}}$. Let Z be any finite subset of Y, that has a non-empty intersection with both Diag($\mathfrak{B}$) and El-diag($\mathfrak{A}$); we can consider Z as given by $Z = \{\xi, \psi\}$ where $\xi \in \text{Diag}(\mathfrak{B})$ and $\psi \in \text{El-diag}(\mathfrak{A})$. Let $c_1, \ldots, c_r$ be the (distinct) constants of $\tau_{\mathfrak{B}} \setminus \tau_{\mathfrak{A}}$ appearing in ξ, and let $x_1, \ldots, x_r$ be fresh variables. Consider the sentence ϕ given by $\phi = \exists x_1 \ldots \exists x_r \xi\, [c_1 \mapsto x_1; \ldots; c_r \mapsto x_r]$, where $c_i \mapsto x_i$ denotes substitution of x_i for c_i, for $1 \leq i \leq r$. Observe that ϕ is a Σ^0_1 sentence of FO($\tau_{\mathfrak{A}}$) and that $\mathfrak{B}_{\mathfrak{A}} \models \phi$. Since $\mathfrak{A} \preceq_1 \mathfrak{B}$, we have that $\mathfrak{A}_{\mathfrak{A}} \models \phi$. Let $a_1, \ldots, a_r$ be the witnesses in $\mathfrak{A}_{\mathfrak{A}}$, of the quantifiers of ϕ corresponding to variables $x_1, \ldots, x_r$. Interpreting the constants $c_1, \ldots, c_r$ as $a_1, \ldots, a_r$ respectively, we see that $(\mathfrak{A}_{\mathfrak{A}}, a_1, \ldots, a_r) \models Z$. Since Z is an arbitrary finite subset of Y, by the compactness theorem, Y is satisfied in a $\tau_{\mathfrak{B}}$-structure $\mathfrak{C}$. The τ-reduct of $\mathfrak{C}$ is the desired structure $\mathfrak{A}'$. □

Proof of Lemma 4.8. Let $\mathfrak{A}$ be a λ-saturated elementary extension of $\mathfrak{A}_1$, for some $\lambda \geq \omega$. We show below the existence of a structure $\mathfrak{B}_2$ such that (i) $\mathfrak{A} \preceq_1 \mathfrak{B}_2$ and (ii) $\mathfrak{B}_1$ is elementarily embeddable in $\mathfrak{B}_2$ via an embedding say f. Let $\mathfrak{B}$ be the image of $\mathfrak{B}_1$ under f; then $\mathfrak{B} \cong \mathfrak{B}_1$ and $\mathfrak{B} \preceq \mathfrak{B}_2$. By Lemma 4.10, there exists a structure $\mathfrak{A}_2$ such that $(\mathfrak{A}, \mathfrak{B}_2, \mathfrak{A}_2)$ is a sandwich, whence $\mathfrak{B}$ is sandwiched by $\mathfrak{A}$. Then $\mathfrak{B}$ is indeed as desired. For our arguments below, we make the following observation, call it (*): If $\mathfrak{B}$ is sandwiched by $\mathfrak{A}$, then every Σ^0_2 sentence true in $\mathfrak{A}$ is also true in $\mathfrak{B}$. This follows simply from Theorem 4.7 by taking T to be the set of all Σ^0_2 sentences that are true in $\mathfrak{A}$, and taking V to be the empty theory.

Let τ be the vocabulary of $\mathfrak{A}$ and $\mathfrak{B}_1$, and let $\tau_{\mathfrak{A}}$ and $\tau_{\mathfrak{B}_1}$ be such that $\tau_{\mathfrak{A}} \cap \tau_{\mathfrak{B}_1} = \tau$. Consider the theory Y given by $Y = S_\Pi(\mathfrak{A}_{\mathfrak{A}}) \cup \text{El-diag}(\mathfrak{B}_1)$, where $S_\Pi(\mathfrak{A}_{\mathfrak{A}})$ denotes the set of all Π^0_1 sentences true in $\mathfrak{A}_{\mathfrak{A}}$. Observe that $S_\Pi(\mathfrak{A}_{\mathfrak{A}})$ is closed under finite conjunctions. Let Z be any non-empty finite subset of Y. If $Z \subseteq S_\Pi(\mathfrak{A}_{\mathfrak{A}})$ or $Z \subseteq \text{El-diag}(\mathfrak{B}_1)$, then Z is clearly satisfiable. Else, $Z = \{\xi, \psi\}$ where $\xi \in S_\Pi(\mathfrak{A}_{\mathfrak{A}})$ and $\psi \in \text{El-diag}(\mathfrak{B}_1)$. Let $c_1, \ldots, c_r$ be the (distinct) constants of $\tau_{\mathfrak{A}} \setminus \tau$ appearing in ξ, and let $x_1, \ldots, x_r$ be fresh variables. Consider the sentence ϕ given by $\phi = \exists x_1 \ldots \exists x_r \xi\, [c_1 \mapsto x_1; \ldots; c_r \mapsto x_r]$, where $c_i \mapsto x_i$ denotes substitution of x_i for c_i, for $1 \leq i \leq r$. Clearly $\mathfrak{A} \models \phi$, whence $\mathfrak{A}_1 \models \phi$. Since $\mathfrak{B}_1$ is sandwiched by $\mathfrak{A}_1$ and ϕ is a Σ^0_2 sentence, it follows from observation (*) above, that $\mathfrak{B}_1 \models \phi$. Let $b_1, \ldots, b_r$ be the witnesses in $\mathfrak{B}_1$ of the quantifiers of ϕ associated with $x_1, \ldots, x_r$. One can now check that if $\mathfrak{R} = \mathfrak{B}_1$, then $(\mathfrak{R}_{\mathfrak{R}}, b_1, \ldots, b_r) \models Z$. Since Z is an arbitrary finite subset of Y, by compactness theorem, Y is satisfiable. Whereby, there exists a τ-structure $\mathfrak{B}_2$ such that (i) $\mathfrak{A} \preceq_1 \mathfrak{B}_2$ and (ii) $\mathfrak{B}_1$ is elementarily embeddable in $\mathfrak{B}_2$. □

We now turn to proving Lemma 4.9. Recall from the previous subsection that for a structure $\mathfrak{A}$ and a k-tuple $\bar{a}$ of $\mathfrak{A}$, we denote by $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$ the Π^0_1-type of $\bar{a}$ in $\mathfrak{A}$. Given theories V and T such that T is $PSC(k)$ modulo V, we say that $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$ *determines a k-crux w.r.t. T modulo V* if it is the case that for any model $\mathfrak{D}$ of V and a k-tuple $\bar{d}$ of $\mathfrak{D}$, if $(\mathfrak{D},\bar{d}) \models \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$, then $\mathfrak{D} \models T$. Since universal formulae are preserved under substructures, it follows that for $\mathfrak{D}$ as just mentioned, the elements of $\bar{d}$ form a k-crux of $\mathfrak{D}$ w.r.t. T modulo V. To prove Lemma 4.9, we need the next result which characterizes when a Π^0_1-type determines a k-crux.

Lemma 4.11 *(Characterizing "crux determination"). Given theories V and T, let T be $PSC(k)$ modulo V for some $k \in \mathbb{N}$. Let $\mathfrak{A} \models V$ and let $\bar{a}$ be a k-tuple of $\mathfrak{A}$. Then $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$ determines a k-crux w.r.t. T modulo V iff $\mathfrak{A} \models T$ and for some $\lambda \geq \omega$, there exists a λ-saturated elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ (hence $\mathfrak{B} \models (V \cup T)$) such that the elements of $\bar{a}$ form a k-crux of $\mathfrak{B}$ w.r.t. T modulo V.*

Proof. 'Only If': Since $\mathfrak{A} \models V$ and $(\mathfrak{A},\bar{a}) \models \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$, we have that $\mathfrak{A} \models T$. Let $\mathfrak{B}$ be a λ-saturated elementary extension of $\mathfrak{A}$, for some $\lambda \geq \omega$; then $\mathfrak{B} \models (V \cup T)$ and $(\mathfrak{B},\bar{a}) \models \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$. Let $\mathfrak{C} \subseteq \mathfrak{B}$ be such that $\mathfrak{C}$ contains $\bar{a}$ and $\mathfrak{C} \models V$. Since universal formulae are preserved under substructures, $(\mathfrak{C},\bar{a}) \models \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$, whence $\mathfrak{C} \models T$. Then, elements of $\bar{a}$ form a k-crux of $\mathfrak{B}$ w.r.t. T modulo V.

'If': Let $\mathfrak{A}$, $\mathfrak{B}$ and $\bar{a}$ be as mentioned in the statement. Consider a model $\mathfrak{D}$ of V and a k-tuple $\bar{d}$ of $\mathfrak{D}$ such that $(\mathfrak{D},\bar{d}) \models \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$. By the downward Löwenheim–Skolem theorem, there exists $\mathfrak{D}_1 \preceq \mathfrak{D}$ such that (i) $\mathfrak{D}_1$ contains $\bar{d}$ and (ii) $|\mathfrak{D}_1| \leq \omega$. Then $(\mathfrak{D}_1,\bar{d}) \models \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$. Now since $\mathfrak{A} \preceq \mathfrak{B}$, we have that $\mathsf{tp}_{\Pi,\mathfrak{B},\bar{a}}(x_1,\ldots,x_k) = \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$. Then every existential sentence that is true in $(\mathfrak{D}_1,\bar{d})$ is also true in $(\mathfrak{B},\bar{a})$. Since $\mathfrak{B}$ is λ-saturated, $(\mathfrak{B},\bar{a})$ is also λ-saturated. Further, since $|\mathfrak{D}_1| \leq \omega$, we have $|(\mathfrak{D}_1,\bar{d})| \leq \omega \leq \lambda$. Then there exists an isomorphic embedding $f : (\mathfrak{D}_1,\bar{d}) \to (\mathfrak{B},\bar{a})$. The image of $(\mathfrak{D}_1,\bar{d})$ under f is a substructure $(\mathfrak{B}_1,\bar{a})$ of $(\mathfrak{B},\bar{a})$ such that $(\mathfrak{D}_1,\bar{d}) \cong (\mathfrak{B}_1,\bar{a})$. Since $\mathfrak{D}_1 \preceq \mathfrak{D}$ and $\mathfrak{D} \models V$, we have $\mathfrak{B}_1 \models V$. Further since the elements of $\bar{a}$ form a k-crux of $\mathfrak{B}$ w.r.t. T modulo V (by assumption), we have $\mathfrak{B}_1 \models T$. Then $\mathfrak{D}_1$, and hence $\mathfrak{D}$, models T, completing the proof. □

Proof of Lemma 4.9. We will assume the vocabulary to be τ. Since $\mathfrak{B}$ is sandwiched by $\mathfrak{A}$, there exist structures $\mathfrak{A}_1$ and $\mathfrak{B}_1$ such that (i) $\mathfrak{B} \preceq \mathfrak{B}_1$ and (ii) $(\mathfrak{A},\mathfrak{B}_1,\mathfrak{A}_1)$ is a sandwich. Let $\mathfrak{D}$ be a μ-saturated elementary extension of $\mathfrak{A}_1$, for some $\mu \geq \omega$. Then $\mathfrak{A} \preceq \mathfrak{D}$. Since $\mathfrak{A}$ models $V \cup T$, so does $\mathfrak{D}$.

Now, given that T is $PSC(k)$ modulo V, there exists a k-crux of $\mathfrak{D}$ w.r.t. T modulo V; let $\bar{d}$ be any k-tuple formed from this k-crux. Consider $\mathsf{tp}_{\mathfrak{D},\bar{d}}(x_1,\ldots,x_k)$, namely the FO-type of $\bar{d}$ in $\mathfrak{D}$. Since $\mathfrak{A} \preceq \mathfrak{D}$, we have $\mathsf{th}(\mathfrak{A}) = \mathsf{th}(\mathfrak{D})$, whence $\mathsf{tp}_{\mathfrak{D},\bar{d}}(x_1,\ldots,x_k)$ is consistent with $\mathsf{th}(\mathfrak{A})$. Since $\mathfrak{A}$ is λ-saturated, we have by definition that $\mathsf{tp}_{\mathfrak{D},\bar{d}}(x_1,\ldots,x_k)$ is realized in $\mathfrak{A}$ by a k-tuple say $\bar{a}$, i.e. $\mathsf{tp}_{\mathfrak{A},\bar{a}}(x_1,\ldots,x_k) = \mathsf{tp}_{\mathfrak{D},\bar{d}}(x_1,\ldots,x_k)$. Then since $\mathfrak{A} \preceq \mathfrak{D}$, it follows that the FO-type of $\bar{a}$ in $\mathfrak{D}$, namely $\mathsf{tp}_{\mathfrak{D},\bar{a}}(x_1,\ldots,x_k)$, is exactly $\mathsf{tp}_{\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$. Whence, $\mathsf{tp}_{\Pi,\mathfrak{D},\bar{a}}(x_1,\ldots,x_k) = \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k) = \mathsf{tp}_{\Pi,\mathfrak{D},\bar{d}}(x_1,\ldots,x_k)$. Now since (i) $\mathfrak{D} \models (V \cup T)$, (ii) $\mathfrak{D}$ is μ-saturated, and (iii) elements of $\bar{d}$ form a k-crux of $\mathfrak{D}$ w.r.t. T modulo V, we have by Lemma 4.11, that $\mathsf{tp}_{\Pi,\mathfrak{D},\bar{d}}(x_1,\ldots,x_k)$, and hence $\mathsf{tp}_{\Pi,\mathfrak{D},\bar{a}}(x_1,\ldots,x_k)$, determines a k-crux w.r.t. T modulo V. Whence the elements of $\bar{a}$ form a k-crux of $\mathfrak{D}$ w.r.t. T modulo V. Since (i) $\mathfrak{B}_1 \subseteq \mathfrak{D}$, (ii) $\mathfrak{B}_1$ contains $\bar{a}$ and (iii) $\mathfrak{B}_1 \models V$ (since $\mathfrak{B} \models V$ and $\mathfrak{B} \preceq \mathfrak{B}_1$), we have by definition of a k-crux, that $\mathfrak{B}_1 \models T$, whence $\mathfrak{B} \models T$. □

We conclude this section with the following lemma.

Lemma 4.12. *There exist theories V and T such that (i) each sentence of T is a Σ^0_2 sentence having exactly one existential quantifier, and (ii) T is not $PSC(k)$ modulo V, for any $k \in \mathbb{N}$.*

Proof. Let $V = \{\forall x \forall y (E(x,y) \to E(y,x))\}$ be the theory that defines exactly all undirected graphs. For $n \geq 1$, let $\varphi_n(x)$ be a formula asserting that x is not a part of a cycle of length n. Explicitly, $\varphi_1(x) = \neg E(x,x)$ and for $n \geq 1$, we have $\varphi_{n+1}(x) = \neg \exists z_1 \ldots \exists z_n \big((\bigwedge_{1 \leq i < j \leq n} z_i \neq z_j) \wedge (\bigwedge_{i=1}^{i=n} (x \neq z_i)) \wedge E(x,z_1) \wedge E(z_n,x) \wedge$

$\bigwedge_{i=1}^{i=n-1} E(z_i, z_{i+1})$). Consider $\chi_n(x) = \bigwedge_{i=1}^{i=n} \varphi_i(x)$ which asserts that x is not a part of any cycle of length $\leq n$. Observe that $\chi_n(x)$ is a universal formula. Also, if $m \leq n$, then $\chi_n(x) \to \chi_m(x)$.

Now consider the theory $T = \{\psi_n \mid n \geq 1\}$, where $\psi_n = \exists x \chi_n(x)$. Each sentence of T is a Σ_2^0 sentence having only one existential variable. We show below that T is not $PSC(k)$ modulo V, for any $k \in \mathbb{N}$.

Consider the infinite graph G given by $G = \bigsqcup_{i\geq 3} C_i$ where C_i is the cycle graph of length i and $\bigsqcup$ denotes disjoint union. Any vertex x of C_i satisfies $\chi_j(x)$ in G, for $j < i$. Then $G \models T$. Now consider any finite set S of vertices of G. Let r be the highest index such that some vertex in S is in the cycle C_r. Consider the subgraph G_1 of G induced by the vertices of all the cycles in G of length $\leq r$. Then no vertex x of G_1 satisfies $\chi_l(x)$ for $l > r$. Then $G_1 \not\models T$, whence S cannot be a k-crux of G w.r.t. T modulo V, for any $k \geq |S|$. Since S was an arbitrary finite subset of G, we conclude that G has no k-crux w.r.t. T modulo V, for any $k \in \mathbb{N}$; whence T is not $PSC(k)$ modulo V, for any $k \in \mathbb{N}$. □

4.3. A conditional refinement of Theorem 4.5

Theorem 4.5, while showing that a $PSC(k)$ theory is always equivalent to a Σ_2^0 theory, does not tell us anything about the maximum number of existential quantifiers that can appear in any sentence of the Σ_2^0 theory. Given Corollary 4.6 that asserts that a $PSC(k)$ sentence is always equivalent to an $\exists^k\forall^*$ sentence, it is natural to ask whether a $PSC(k)$ theory is equivalent to a theory of $\exists^k\forall^*$ sentences. We answer this question in the affirmative – thereby refining Theorem 4.5 – under the hypothesis that every model of a $PSC(k)$ theory always contains a k-tuple whose Π_1^0-type determines a k-crux. The technique of our proof is as presented below.

1. We first define a variant of $PSC(k)$, denoted $PSC_{var}(k)$, into whose definition we build the hypothesis.
2. We then show that $PSC_{var}(k)$ theories are equivalent to theories of $\exists^k\forall^*$ sentences. This is done in the following two steps:
   - "Going up": We give a characterization of $PSC_{var}(k)$ theories in terms of sentences of a special infinitary logic (Lemma 4.22).
   - "Coming down": We provide a translation of sentences of the aforesaid infinitary logic, into their equivalent FO theories, whenever these sentences define elementary classes of structures (Proposition 4.23). The FO theories are obtained from suitable *finite approximations* of the infinitary sentences, and turn out to be theories of $\exists^k\forall^*$ sentences.
3. We hypothesize that $PSC_{var}(k)$ theories are no different from $PSC(k)$ theories to get the refinement of Theorem 4.5, referred to at the outset (Theorem 4.20 and Remark 4.21). To show that this hypothesis is well-motivated, we define a variant of $PCE(k)$, denoted $PCE_{var}(k)$, that is dual to $PSC_{var}(k)$. We show that $PCE_{var}(k)$ coincides with $PCE(k)$ for theories, and use this to conclude that $PSC_{var}(k)$ coincides with $PSC(k)$ for sentences (Lemma 4.18).

Throughout the section, whenever V and T are clear from the context, we skip mentioning the qualifier 'w.r.t. T modulo V' for a k-crux, if T is $PSC(k)$ modulo V. If T is $PSC(k)$ modulo V and $\mathfrak{A}$ is a model of $V \cup T$, then we abuse terminology and call a k-tuple $\bar{a}$ of $\mathfrak{A}$ as a k-crux of $\mathfrak{A}$, if the underlying set of elements of $\bar{a}$ forms a k-crux of $\mathfrak{A}$. Before we present the definitions of $PSC_{var}(k)$ and $PCE_{var}(k)$, we first define the notions of 'distinguished k-crux' and 'k-ary cover of a structure $\mathfrak{A}$ in an elementary extension of $\mathfrak{A}$'.

Definition 4.13. Suppose T is $PSC(k)$ modulo V for theories T and V. Given a model $\mathfrak{A}$ of $V \cup T$, we call a k-tuple $\bar{a}$ of $\mathfrak{A}$ a *distinguished k-crux* of $\mathfrak{A}$, if for some $\lambda \geq \omega$, there is a λ-saturated elementary extension $\mathfrak{A}^+$ of $\mathfrak{A}$ (whence $\mathfrak{A}^+ \models (V \cup T)$) such that $\bar{a}$ is a k-crux of $\mathfrak{A}^+$ (whence $\bar{a}$ is also a k-crux of $\mathfrak{A}$).

Remark 4.14. From Lemma 4.11, we can see that $\bar{a}$ is a distinguished k-crux of $\mathfrak{A}$ iff $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(x_1,\ldots,x_k)$ determines a k-crux.

Definition 4.15. Let $\mathfrak{A}$ be a structure and $\mathfrak{A}^+$ be an elementary extension of $\mathfrak{A}$. A non-empty collection R of substructures of $\mathfrak{A}^+$ is said to be a *k-ary cover of $\mathfrak{A}$ in $\mathfrak{A}^+$* if for every k-tuple $\bar{a}$ of elements of $\mathfrak{A}$, there exists a structure in R containing $\bar{a}$.

Observe that the notion of a 'k-ary cover of $\mathfrak{A}$' as in Definition 3.5 corresponds to the notion in Definition 4.15 above, with $\mathfrak{A}^+$ being the same as $\mathfrak{A}$.

Definition 4.16. Let V and T be theories.

1. We say T is $PSC_{var}(k)$ modulo V if T is $PSC(k)$ modulo V and every model of $V \cup T$ contains a distinguished k-crux.
2. We say T is $PCE_{var}(k)$ modulo V if for every model $\mathfrak{A}$ of V, there exists a λ-saturated elementary extension $\mathfrak{A}^+$ of $\mathfrak{A}$ for some $\lambda \geq \omega$, such that for every collection R of models of $V \cup T$, if R is a k-ary cover of $\mathfrak{A}$ in $\mathfrak{A}^+$, then $\mathfrak{A} \models T$.

If $\phi(\bar{x})$ and $T(\bar{x})$ are respectively a formula and a theory, each of whose free variables are among $\bar{x}$, then for a theory V, the notions of '$\phi(\bar{x})$ is $PSC_{var}(k)$ (resp. $PCE_{var}(k)$) modulo V' and '$T(\bar{x})$ is $PSC_{var}(k)$ (resp. $PCE_{var}(k)$) modulo V' are defined similar to corresponding notions for $PSC(k)$ (resp. $PCE(k)$). The following duality is easy to see.

Lemma 4.17 *($PSC_{var}(k)$–$PCE_{var}(k)$ duality). Given a theory V, a formula $\phi(\bar{x})$ is $PSC_{var}(k)$ modulo V iff $\neg\phi(\bar{x})$ is $PCE_{var}(k)$ modulo V.*

Towards the central result of this subsection, we first show the following.

Lemma 4.18. *Given a theory V, each of the following holds.*

1. *A formula $\phi(\bar{x})$ is $PSC(k)$ modulo V iff $\phi(\bar{x})$ is $PSC_{var}(k)$ modulo V.*
2. *A theory $T(\bar{x})$ is $PCE(k)$ modulo V iff $T(\bar{x})$ is $PCE_{var}(k)$ modulo V.*

Proof. We show below the following equivalence, call it (†): A theory $T(\bar{x})$ is $PCE_{var}(k)$ modulo V iff $T(\bar{x})$ is equivalent modulo V to a theory of $\forall^k\exists^*$ formulae, all of whose free variables are among $\bar{x}$. Then Part (2) of this lemma follows from (†) and Theorem 4.1. Part (1) of the lemma in turn follows from Part (2) and the dualities given by Lemma 3.9 and Lemma 4.17.

Given the observation following Definition 4.15, we can prove the 'Only if' direction of (†) in a manner identical to the proof of the 'Only if' direction of Theorem 4.1. The proof of the 'If' direction of (†) is also nearly the same as that of the 'If' direction of Theorem 4.1; we present this proof below for completeness. It suffices to give the proof for theories without free variables.

Let T be equivalent modulo V to a theory of $\forall^k\exists^*$ sentences. Given a model $\mathfrak{A}$ of V, let $\mathfrak{A}^+$ be a λ-saturated elementary extension of $\mathfrak{A}$, for some $\lambda \geq \omega$. Let R be a collection of models of $V \cup T$ that forms a k-ary cover of $\mathfrak{A}$ in $\mathfrak{A}^+$. We show that $\mathfrak{A} \models T$. Consider $\varphi \in T$; let $\varphi = \forall^k\bar{x}\psi(\bar{x})$ for a Σ^0_1 formula $\psi(\bar{x})$, and let $\bar{a}$ be a k-tuple of $\mathfrak{A}$. Since R is a k-ary cover of $\mathfrak{A}$ in $\mathfrak{A}^+$, there exists $\mathfrak{B}_{\bar{a}} \in R$ such that $\mathfrak{B}_{\bar{a}}$ contains $\bar{a}$. Since $\mathfrak{B}_{\bar{a}} \models (V \cup T)$, we have $\mathfrak{B}_{\bar{a}} \models \varphi$ and hence $(\mathfrak{B}_{\bar{a}}, \bar{a}) \models \psi(\bar{x})$. Since $\psi(\bar{x})$ is a Σ^0_1 formula and $\mathfrak{B}_{\bar{a}} \subseteq \mathfrak{A}^+$, we have $(\mathfrak{A}^+, \bar{a}) \models \psi(\bar{x})$, whence $(\mathfrak{A}, \bar{a}) \models \psi(\bar{x})$ since $\mathfrak{A} \preceq \mathfrak{A}^+$. Since $\bar{a}$ is arbitrary, $\mathfrak{A} \models \varphi$, and since φ is an arbitrary sentence of T, we have $\mathfrak{A} \models T$. □

Motivated by Lemma 4.18, we put forth the hypothesis below.

Hypothesis 4.19. If V and $T(\bar{x})$ are theories, then $T(\bar{x})$ is $PSC(k)$ modulo V iff $T(\bar{x})$ is $PSC_{var}(k)$ modulo V.

In other words, Hypothesis 4.19 states that if a theory T is $PSC(k)$ modulo V, then every model of $V \cup T$ contains a distinguished k-crux. We now formally state the central result of this subsection.

Theorem 4.20. *Given theories V and $T(\bar{x})$, suppose $T(\bar{x})$ is $PSC_{var}(k)$ modulo V. Then $T(\bar{x})$ is equivalent modulo V to a theory of Σ^0_2 formulae, all of whose free variables are among $\bar{x}$, and all of which have k existential quantifiers.*

Remark 4.21. It follows from Theorem 4.20 that if $T(\bar{x})$ is $PSC(k)$ modulo V, then assuming Hypothesis 4.19 holds, $T(\bar{x})$ is equivalent modulo V to a theory of Σ^0_2 formulae, all of whose free variables are among $\bar{x}$, and all of which have k existential quantifiers. This gives us a *conditional refinement of Theorem 4.5* seen in the previous subsection.

We devote the rest of this section to proving Theorem 4.20. We first introduce some notation and terminology. These are adapted versions of similar notation and terminology introduced in [9] and [10]. Given a class $\mathcal{F}$ of formulae and $k \geq 0$, denote by $\left[\exists^k \bigwedge\right] \mathcal{F}$ the class of infinitary formulae $\Phi(\bar{x})$ of the form $\exists y_1 \ldots \exists y_k \bigwedge_{i \in I} \psi_i(y_1, \ldots, y_k, \bar{x})$ where I is an index set (of arbitrary cardinality) and for each $i \in I$, ψ_i is a formula of $\mathcal{F}$, whose free variables are among $y_1, \ldots, y_k, \bar{x}$. Let $[\exists^* \bigwedge] \mathcal{F} = \bigcup_{k \geq 0} \left[\exists^k \bigwedge\right] \mathcal{F}$. Observe that $\mathcal{F} \subseteq [\exists^* \bigwedge] \mathcal{F}$. For each $j \in \mathbb{N}$, let $[\exists^* \bigwedge]^j \mathcal{F} = [\exists^* \bigwedge]\ [\exists^* \bigwedge]^{j-1} \mathcal{F}$, where $[\exists^* \bigwedge]^0 \mathcal{F} = \mathcal{F}$. Let $[\exists^* \bigwedge]^* \mathcal{F} = \bigcup_{j \geq 0} [\exists^* \bigwedge]^j \mathcal{F}$. Finally, let $[\bigvee] \mathcal{F}$ denote arbitrary disjunctions of formulae of $\mathcal{F}$. It is easy to see that $\mathcal{F} \subseteq [\bigvee] \mathcal{F}$.

Let $\Phi(\bar{x})$ be a formula of $[\bigvee]\, [\exists^* \bigwedge]^*$ FO, where FO denotes the class of all first order formulae. We define below, the set $\mathcal{A}(\Phi)(\bar{x})$ of *finite approximations* of $\Phi(\bar{x})$. Let $\subseteq_f$ denote 'finite subset of'.

1. If $\Phi(\bar{x}) \in$ FO, then $\mathcal{A}(\Phi)(\bar{x}) = \{\Phi(\bar{x})\}$.
2. If $\Phi(\bar{x}) = \exists^k \bar{y} \bigwedge_{i \in I} \Psi_i(\bar{x}, \bar{y})$ for $k \geq 0$ and some index set I, then $\mathcal{A}(\Phi)(\bar{x}) = \{\exists^k \bar{y} \bigwedge_{i \in I_1} \gamma_i(\bar{x}, \bar{y}) \mid \gamma_i(\bar{x}, \bar{y}) \in \mathcal{A}(\Psi_i)(\bar{x}, \bar{y}),\ I_1 \subseteq_f I\}$.
3. If $\Phi(\bar{x}) = \bigvee_{i \in I} \Psi_i(\bar{x})$, then $\mathcal{A}(\Phi)(\bar{x}) = \{\bigvee_{i \in I_1} \gamma_i(\bar{x}) \mid \gamma_i(\bar{x}) \in \mathcal{A}(\Psi_i)(\bar{x}),\ I_1 \subseteq_f I\}$.

Our proof of Theorem 4.20 is in two parts. The first part, namely the "going up" part as alluded to in the beginning of this subsection, gives a characterization of $PSC_{var}(k)$ theories in terms of the formulae of $[\bigvee] \left[\exists^k \bigwedge\right] \Pi^0_1$, where Π^0_1 is, as usual, the class of all prenex FO formulae having only universal quantifiers.

Lemma 4.22. *Let V and $T(\bar{x})$ be given theories. Then $T(\bar{x})$ is $PSC_{var}(k)$ modulo V iff $T(\bar{x})$ is equivalent modulo V to a formula of $[\bigvee] \left[\exists^k \bigwedge\right] \Pi^0_1$, whose free variables are among $\bar{x}$.*

The second part of the proof of Theorem 4.20, namely the "coming down" part, consists of getting FO theories equivalent to the formulae of $[\bigvee] \left[\exists^k \bigwedge\right] \Pi^0_1$, whenever the latter define elementary classes of structures. In fact, we show a more general result as we now describe. Given a theory V, we say that a formula $\Phi(x_1, \ldots, x_k)$ of $[\bigvee]\, [\exists^* \bigwedge]^*$ FO (over a vocabulary say τ) *defines an elementary class modulo V* if the sentence (over the vocabulary τ_k) obtained by substituting fresh and distinct constants $c_1, \ldots, c_k$ for $x_1, \ldots, x_k$ in $\Phi(x_1, \ldots, x_k)$, defines an elementary class (of τ_k-structures) modulo V. The result below characterizes formulae of $[\bigvee]\, [\exists^* \bigwedge]^*$ FO that define elementary classes, in terms of the finite approximations of these formulae.

Proposition 4.23. *Let $\Phi(\bar{x})$ be a formula of $[\bigvee]\,[\exists^* \bigwedge]^*$ FO and V be a given theory. Then $\Phi(\bar{x})$ defines an elementary class modulo V iff $\Phi(\bar{x})$ is equivalent modulo V to a countable subset of $\mathcal{A}(\Phi)(\bar{x})$.*

The above results prove Theorem 4.20 as follows.

Proof of Theorem 4.20. For any formula $\Phi(\bar{x})$ of $[\bigvee]\,[\exists^k \bigwedge]\,\Pi^0_1$, each formula of the set $\mathcal{A}(\Phi)(\bar{x})$ can be seen to be equivalent to an $\exists^k\forall^*$ formula whose free variables are among $\bar{x}$. The result then follows from Lemma 4.22 and Proposition 4.23. □

Before we provide the proofs of Lemma 4.22 and Proposition 4.23, we state the following compactness result for formulae of $[\exists^* \bigwedge]^*$ FO, that we prove enroute proving Proposition 4.23.

Lemma 4.24. *Let $\Phi(\bar{x})$ be a formula of $[\exists^* \bigwedge]^*$ FO. If every formula of $\mathcal{A}(\Phi)(\bar{x})$ is satisfiable modulo a theory V, then $\Phi(\bar{x})$ is satisfiable modulo V.*

Observe that the standard compactness theorem for FO is a special case of the above result: Given an FO theory $T(\bar{x})$, let $\Phi(\bar{x})$ be the formula of $[\exists^* \bigwedge]^*$ FO given by $\Phi(\bar{x}) = \bigwedge T(\bar{x})$. Then every formula of $\mathcal{A}(\Phi)(\bar{x})$ is equivalent to a finite subset of $T(\bar{x})$ and vice-versa.

Remark 4.25. The formulas of $[\exists^* \bigwedge]^*$ FO are special kinds of "conjunctive formulas", where the latter are as defined in [10]. The paper [10] gives a generalization of the compactness theorem by proving a compactness result for conjunctive formulas, whose statement is similar to that of Lemma 4.24. However, Lemma 4.24 does not follow from this result of [10] because the set of finite approximations of sentences $\Phi(\bar{x})$ of $[\exists^* \bigwedge]^*$ FO, as defined in [10], is semantically strictly larger than the set $\mathcal{A}(\Phi)(\bar{x})$ that we have defined. Further, the techniques that we use in proving Lemma 4.24 are much different from those used in [10] for proving the compactness result for conjunctive formulas.

In the remainder of this subsection, we give proofs for Lemmas 4.22 and 4.24, and Proposition 4.23. For Lemma 4.22 and Proposition 4.23, it suffices to give the proofs only for theories/formulae without free variables. For Lemma 4.24, we give the proof for formulae with free variables, since the proof is by induction on the structure of the formulae.

Proof of Lemma 4.22. If: Let T be equivalent modulo V to the sentence $\Phi = \bigvee_{i\in I} \exists^k \bar{y}_i \bigwedge Y_i(\bar{y}_i)$, where I is an index set and for each $i \in I$, Y_i is a set of Π^0_1 formulae, all of whose free variables are among $\bar{y}_i$. Then given a model $\mathfrak{A}$ of $V \cup T$, there exist $i \in I$ and $\bar{a}$ in $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{a}) \models \bigwedge Y_i(\bar{y}_i)$. Let $\mathfrak{A}^+$ be a λ-saturated elementary extension of $\mathfrak{A}$, for some $\lambda \geq \omega$. Then $(\mathfrak{A}^+, \bar{a}) \models \bigwedge Y_i(\bar{y}_i)$. Whence for each $\mathfrak{B} \subseteq \mathfrak{A}^+$ such that $\mathfrak{B}$ contains $\bar{a}$, $(\mathfrak{B}, \bar{a}) \models \bigwedge Y_i(\bar{y}_i)$, and hence $\mathfrak{B} \models \Phi$. Since Φ is equivalent to T modulo V, we have $\bar{a}$ as a distinguished k-crux of $\mathfrak{A}$.

Only If: Suppose T is $PSC_{var}(k)$ modulo V. Given a model $\mathfrak{A}$ of $V \cup T$, let $\mathsf{Dist\text{-}k\text{-}cruxes}(\mathfrak{A})$ be the (non-empty) set of all distinguished k-cruxes of $\mathfrak{A}$. Consider the sentence $\Phi = \bigvee_{\mathfrak{A}\models V\cup T,\ \bar{a}\in \mathsf{Dist\text{-}k\text{-}cruxes}(\mathfrak{A})} \exists^k \bar{x} \bigwedge \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(\bar{x})$, where $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(\bar{x})$ is the Π^0_1-type of $\bar{a}$ in $\mathfrak{A}$. We show that T is equivalent to Φ modulo V. That T implies Φ modulo V is obvious from the definition of Φ. Towards the converse, suppose $\mathfrak{B} \models \{\Phi\} \cup V$. Then for some model $\mathfrak{A}$ of $V \cup T$, some distinguished k-crux $\bar{a}$ of $\mathfrak{A}$, and for some k-tuple $\bar{b}$ of $\mathfrak{B}$, we have $(\mathfrak{B}, \bar{b}) \models \mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(\bar{x})$. By Remark 4.14, $\mathsf{tp}_{\Pi,\mathfrak{A},\bar{a}}(\bar{x})$ determines a k-crux, whence $\mathfrak{B} \models T$. □

To prove Lemma 4.24 and Proposition 4.23, we need the auxiliary lemmas below.

Lemma 4.26. *For $j \in \mathbb{N}$, let $T(\bar{x})$ be a set of formulae of $[\exists^* \bigwedge]^j$ FO, all of whose free variables are among $\bar{x}$. If every finite subset of $T(\bar{x})$ is satisfiable modulo a theory V, then $T(\bar{x})$ is satisfiable modulo V.*

Proof. We prove the statement by induction on j. The base case of $j = 0$ is the standard compactness theorem. As induction hypothesis, suppose the statement is true for j. For the inductive step, consider a set $T(\bar{x}) = \{\Phi_i(\bar{x}) \mid i \in I\}$ of $[\exists^* \bigwedge]^{j+1}$ FO formulae, all of whose free variables are among $\bar{x}$, and suppose every finite subset of $T(\bar{x})$ is satisfiable modulo V. Let $\Phi_i(\bar{x}) = \exists \bar{y}_i \bigwedge T_i(\bar{x}, \bar{y}_i)$ where $T_i(\bar{x}, \bar{y}_i)$ is a set of formulae of $[\exists^* \bigwedge]^j$ FO. Assume for $i, j \in I$ and $i \neq j$, that $\bar{y}_i$ and $\bar{y}_j$ have no common variables. We show that the set Y of $[\exists^* \bigwedge]^j$ FO formulae given by $Y = \bigcup_{i \in I} T_i$ is satisfiable modulo V; then so is $T(\bar{x})$.

By the induction hypothesis, it suffices to show that every finite subset Z of Y is satisfiable modulo V. Let $Z(\bar{x}, \bar{y}_{i_1}, \ldots, \bar{y}_{i_n}) = \bigcup_{r=1}^{r=n} Z_r(\bar{x}, \bar{y}_{i_r})$, where $n > 0$, $Z_r(\bar{x}, \bar{y}_{i_r}) \subseteq_f T_{i_r}(\bar{x}, \bar{y}_{i_r})$ and $i_r \in I$, for each $r \in \{1, \ldots, n\}$. The subset $\{\Phi_{i_r}(\bar{x}) \mid r \in \{1, \ldots, n\}\}$ of $T(\bar{x})$ is satisfiable modulo V by assumption, whence for some model $\mathfrak{A}$ of V, and interpretations $\bar{a}$ of $\bar{x}$ and $\bar{b}_{i_r}$ of $\bar{y}_{i_r}$, we have that $\bigcup_{r=1}^{r=n} T_{i_r}(\bar{x}, \bar{y}_{i_r})$ is satisfied in $(\mathfrak{A}, \bar{a}, \bar{b}_{i_1}, \ldots, \bar{b}_{i_n})$; then $(\mathfrak{A}, \bar{a}, \bar{b}_{i_1}, \ldots, \bar{b}_{i_n}) \models Z(\bar{x}, \bar{y}_{i_1}, \ldots, \bar{y}_{i_n})$. □

Lemma 4.27. *Let $\Phi(\bar{x})$ be a formula of $[\exists^* \bigwedge]^*$ FO. If $(\mathfrak{A}, \bar{a}) \models \Phi(\bar{x})$, then $(\mathfrak{A}, \bar{a}) \models \xi(\bar{x})$ for every formula $\xi(\bar{x})$ of $\mathcal{A}(\Phi)(\bar{x})$.*

Proof. The proof is by induction. The statement is trivial for formulae of FO $= [\exists^* \bigwedge]^0$ FO. Assume the statement holds for $[\exists^* \bigwedge]^j$ FO formulae. Consider an $[\exists^* \bigwedge]^{j+1}$ FO formula $\Phi(\bar{x}) = \exists^n \bar{y} \bigwedge_{i \in I} \Psi_i(\bar{x}, \bar{y})$, where $\Psi_i(\bar{x}, \bar{y}) \in [\exists^* \bigwedge]^j$ FO for each $i \in I$. Consider a formula $\xi(\bar{x})$ of $\mathcal{A}(\Phi)(\bar{x})$; then $\xi(\bar{x}) = \exists^n \bar{y} \bigwedge_{i \in I_1} \gamma_i(\bar{x}, \bar{y})$, for some $I_1 \subseteq_f I$ and $\gamma_i(\bar{x}, \bar{y}) \in \mathcal{A}(\Psi_i)(\bar{x}, \bar{y})$ for each $i \in I_1$. Since $(\mathfrak{A}, \bar{a}) \models \Phi(\bar{x})$, there is an n-tuple $\bar{b}$ from $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{a}, \bar{b}) \models \Psi_i(\bar{x}, \bar{y})$ for each $i \in I_1$. By induction hypothesis, $(\mathfrak{A}, \bar{a}, \bar{b}) \models \gamma_i(\bar{x}, \bar{y})$ for each $i \in I_1$; then $(\mathfrak{A}, \bar{a}) \models \xi(\bar{x})$. □

Proof of Lemma 4.24. The proof proceeds by induction. The statement trivially holds for formulae of FO $= [\exists^* \bigwedge]^0$ FO. Assume the statement is true for formulae of $[\exists^* \bigwedge]^j$ FO. Consider a formula $\Phi(\bar{x})$ of $[\exists^* \bigwedge]^{j+1}$ FO given by $\Phi(\bar{x}) = \exists \bar{y} \bigwedge_{i \in I} \Psi_i(\bar{x}, \bar{y})$, where $\Psi_i(\bar{x}, \bar{y})$ is a formula of $[\exists^* \bigwedge]^j$ FO for each $i \in I$. We show that every finite subset of $T(\bar{x}, \bar{y}) = \{\Psi_i(\bar{x}, \bar{y}) \mid i \in I\}$ is satisfiable modulo V. Then by Lemma 4.26, $T(\bar{x}, \bar{y})$ is satisfiable modulo V; whence so is $\Phi(\bar{x})$.

Let I_1 be a finite subset of I. For $i \in I_1$, consider the formula $\Psi_i(\bar{x}, \bar{y})$ of $T(\bar{x}, \bar{y})$; it is given by $\Psi_i(\bar{x}, \bar{y}) = \exists \bar{z}_i \bigwedge Z_i(\bar{x}, \bar{y}, \bar{z}_i)$ where $Z_i(\bar{x}, \bar{y}, \bar{z}_i)$ is a set of formulas of $[\exists^* \bigwedge]^{j-1}$ FO. Let $\bar{z} = (\bar{z}_i)_{i \in I_1}$ be the tuple of all the variables of the $\bar{z}_i$s, for i ranging over I_1. Assume without loss of generality that for $i_1, i_2 \in I$ such that $i_1 \neq i_2$, none of the variables of $\bar{z}_{i_1}$ appear in Ψ_{i_2}. Consider the formula $\Psi(\bar{x}, \bar{y})$ of $[\exists^* \bigwedge]^j$ FO given by $\Psi(\bar{x}, \bar{y}) = \exists \bar{z} \bigwedge \left(\bigcup_{i \in I_1} Z_i(\bar{x}, \bar{y}, \bar{z}_i) \right)$. It is easy to verify that $\Psi(\bar{x}, \bar{y})$ is equivalent (over all structures) to $\{\Psi_i(\bar{x}, \bar{y}) \mid i \in I_1\}$. We now show that the latter is satisfiable modulo V by showing that the former is satisfiable modulo V – this in turn is done by showing that every formula in $\mathcal{A}(\Psi)(\bar{x}, \bar{y})$ is satisfiable modulo V, and then applying the induction hypothesis mentioned at the outset.

Let $\gamma(\bar{x}, \bar{y})$ be an arbitrary formula of $\mathcal{A}(\Psi)(\bar{x}, \bar{y})$. Then $\gamma(\bar{x}, \bar{y})$ is of the form $\exists \bar{z} \bigwedge_{i \in I_2} \bigwedge_{l \in \{1, \ldots, n_i\}} \alpha_{i,l}(\bar{x}, \bar{y}, \bar{z}_i)$, where $I_2 \subseteq I_1$, and for each $i \in I_2$, we have $n_i \geq 1$, $\alpha_{i,l}(\bar{x}, \bar{y}, \bar{z}_i) \in \mathcal{A}(\beta_{i,l})(\bar{x}, \bar{y}, \bar{z}_i)$, and $\{\beta_{i,1}(\bar{x}, \bar{y}, \bar{z}_i), \ldots, \beta_{i,n_i}(\bar{x}, \bar{y}, \bar{z}_i)\} \subseteq_f Z_i(\bar{x}, \bar{y}, \bar{z}_i)$. It is easy to see that $\gamma(\bar{x}, \bar{y})$ is equivalent to the formula $\bigwedge_{i \in I_2} \gamma_i(\bar{x}, \bar{y})$ where $\gamma_i(\bar{x}, \bar{y}) = \exists \bar{z}_i \bigwedge_{l \in \{1, \ldots, n_i\}} \alpha_{i,l}(\bar{x}, \bar{y}, \bar{z}_i)$. Observe now that $\gamma_i(\bar{x}, \bar{y}) \in \mathcal{A}(\Psi_i)(\bar{x}, \bar{y})$, whence $\exists \bar{y} \bigwedge_{i \in I_2} \gamma_i(\bar{x}, \bar{y}) \in \mathcal{A}(\Phi)(\bar{x})$. By assumption, every formula of $\mathcal{A}(\Phi)(\bar{x})$ is satisfiable modulo V; then so are $\exists \bar{y} \bigwedge_{i \in I_2} \gamma_i(\bar{x}, \bar{y})$ and $\gamma(\bar{x}, \bar{y})$. □

Proof of Proposition 4.23. It suffices to show just the 'Only if' direction of the result. Hence, consider a sentence Φ of $[\bigvee] [\exists^* \bigwedge]^*$ FO given by $\Phi = \bigvee_{i \in I} \Psi_i$ where $\Psi_i \in [\exists^* \bigwedge]^*$ FO. Let $\mathcal{B} = \prod_{i \in I} \mathcal{A}(\Psi_i)$ where $\prod$ denotes Cartesian product. We now show the following equivalences modulo V:

$$\Phi \leftrightarrow \bigvee_{i \in I} \bigwedge_{\gamma \in \mathcal{A}(\Psi_i)} \gamma \tag{1}$$

$$\leftrightarrow \bigwedge_{(\gamma_i) \in \mathcal{B}} \bigvee_{i \in I} \gamma_i \tag{2}$$

In equivalence Eq. (2) above, (γ_i) denotes a sequence in $\mathcal{B}$. Let $\mathcal{P}_{fin}(I)$ be the set of all finite subsets of I. We finally show the existence of a function $g : \mathcal{B} \to \mathcal{P}_{fin}(I)$ that gives the following equivalence

$$\Phi \leftrightarrow \bigwedge_{(\gamma_i) \in \mathcal{B}} \bigvee_{j \in g((\gamma_i))} \gamma_j \tag{3}$$

Observe that each disjunction in the RHS of Eq. (3) is a sentence of $\mathcal{A}(\Phi)$. Observe also that instead of ranging over all of $\mathcal{B}$ in the RHS of Eq. (3) above, we can range over only a countable subset of $\mathcal{B}$, since the number of FO sentences over a finite vocabulary is countable. We now show the above equivalences to complete the proof. The equivalence Eq. (2) is obtained by applying the standard distributivity laws for conjunctions and disjunctions, to the sentence in the RHS of Eq. (1).

Proof of Eq. (1): Let $\Gamma = \bigvee_{i \in I} \bigwedge_{\gamma \in \mathcal{A}(\Psi_i)} \gamma$. Let $\mathfrak{A}$ be a model of V such that $\mathfrak{A} \models \Phi$. Then $\mathfrak{A} \models \Psi_i$ for some $i \in I$. By Lemma 4.27, we have $\mathfrak{A} \models \mathcal{A}(\Psi_i)$, whence $\mathfrak{A} \models \Gamma$. Thus Φ implies Γ modulo V. Towards the converse, let $\mathfrak{A}$ be a model of V such that $\mathfrak{A} \models \Gamma$. Then $\mathfrak{A} \models \mathcal{A}(\Psi_i)$ for some $i \in I$. Let $\Psi = \bigwedge \left(\mathsf{th}(\mathfrak{A}) \cup \{\Psi_i\}\right)$, where $\mathsf{th}(\mathfrak{A})$ denotes the theory of $\mathfrak{A}$. It is easy to see that $\mathfrak{A} \models \mathcal{A}(\Psi)$ because any sentence γ in $\mathcal{A}(\Psi)$ is given by either $\gamma = \bigwedge Z$ or $\gamma = \gamma_i \wedge \bigwedge Z$, where $Z \subseteq_f \mathsf{th}(\mathfrak{A})$ and $\gamma_i \in \mathcal{A}(\Psi_i)$. Also observe that $\Psi \in [\exists^* \bigwedge]^*$ FO; then since every sentence of $\mathcal{A}(\Psi)$ is satisfiable modulo V, it follows from Lemma 4.24 that Ψ is satisfied in a model of V, say $\mathfrak{B}$. Then (i) $\mathfrak{B} \equiv \mathfrak{A}$ and (ii) $\mathfrak{B} \models \Psi_i$ whence $\mathfrak{B} \models \Phi$. Since Φ defines an elementary class modulo V, we have $\mathfrak{A} \models \Phi$.

Proof of Eq. (3): We show the following result, call it (‡): If T, S and V are FO theories such that $T \to \bigvee S$ modulo V, then T implies $\bigvee S'$ modulo V for some finite subset S' of S. Then Eq. (3) follows from Eq. (2) as follows. By Eq. (2), we have $\Phi \to \bigvee_{i \in I} \gamma_i$ modulo V for each sequence (γ_i) of $\mathcal{B}$ (recall that $\mathcal{B} = \prod_{i \in I} \mathcal{A}(\Psi_i)$). Then by (‡), $\Phi \to \bigvee_{i \in I_1} \gamma_i$ for some $I_1 \subseteq_f I$. Defining $g((\gamma_i)) = I_1$, we get the forward direction of Eq. (3). The backward direction of Eq. (3) is trivial from Eq. (2) and the fact that $\bigvee_{i \in I_1} \gamma_i \to \bigvee_{i \in I} \gamma_i$. We now show (‡).

Since $T \to \bigvee S$ modulo V, we have that $T \cup \{\neg\alpha \mid \alpha \in S\}$ is unsatisfiable modulo V. Then by compactness theorem, $T \cup \{\neg\alpha \mid \alpha \in S'\}$ is unsatisfiable modulo V, for some finite subset S' of S. Whereby, $T \to \bigvee S'$ modulo V. □

5. Preservation properties in terms of finite cruxes and finitary covers

In this section, we present natural generalizations of the $PSC(k)$ and $PCE(k)$ properties in which, rather than insisting on bounded sized cruxes and bounded arity covers, we allow finite sized cruxes and finitary covers; the sizes of the cruxes and the arities of the covers are allowed to be unbounded across the structures of the class considered.

We first define the notion of *finitary covered extensions*.

Definition 5.1. A structure $\mathfrak{A}$ is called a *finitary covered extension* of a collection R of structures if (i) $\mathfrak{A}$ is an extension of R, (ii) for each *finite* subset C of the universe of $\mathfrak{A}$, there is a structure in R containing C. We call R a *finitary cover* of $\mathfrak{A}$.

Observe that, in contrast to k-ary covered extensions, if $\mathfrak{A}$ is a finitary covered extension of R, then R is necessarily non-empty. Further, $\mathfrak{A}$ must be unique such since all predicates and function symbols have finite arity.

Definition 5.2. Let $\mathcal{U}$ be a class of structures and $\mathcal{S}$ be a subclass of $\mathcal{U}$.

1. We say $\mathcal{S}$ is *preserved under substructures modulo finite cruxes over* $\mathcal{U}$, abbreviated $\mathcal{S}$ is PSC_f over $\mathcal{U}$, if for each structure $\mathfrak{A} \in \mathcal{S}$, there is a finite subset C of the universe of $\mathfrak{A}$ such that, if $\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B}$ contains C and $\mathfrak{B} \in \mathcal{U}$, then $\mathfrak{B} \in \mathcal{S}$. The set C is called a *crux (or a finite crux) of* $\mathfrak{A}$ *w.r.t.* $\mathcal{S}$ *over* $\mathcal{U}$.
2. We say $\mathcal{S}$ is *preserved under finitary covered extensions over* $\mathcal{U}$, abbreviated $\mathcal{S}$ *is* PCE_f *over* $\mathcal{U}$, if for every collection R of structures of $\mathcal{S}$, if $\mathfrak{A}$ is a finitary covered extension of R and $\mathfrak{A} \in \mathcal{U}$, then $\mathfrak{A} \in \mathcal{S}$.

If $\phi(\bar{x})$ and $T(\bar{x})$ are respectively a formula and a theory, each of whose free variables are among $\bar{x}$, then given a theory V, the notions of '$\phi(\bar{x})$ is PSC_f (resp. PCE_f) modulo V' and '$T(\bar{x})$ is PSC_f (resp. PCE_f) modulo V' are defined similar to corresponding notions for $PSC(k)$ (resp. $PCE(k)$). Analogous to the results in Sections 3 and 4, we have the following results for PSC_f and PCE_f. The proofs are similar to the corresponding results for $PSC(k)$ and $PCE(k)$, and are hence skipped.

Lemma 5.3 *(PSC_f–PCE_f duality). Let $\mathcal{U}$ be a class of structures, $\mathcal{S}$ be a subclass of $\mathcal{U}$ and $\overline{\mathcal{S}}$ be the complement of $\mathcal{S}$ in $\mathcal{U}$. Then $\mathcal{S}$ is PSC_f over $\mathcal{U}$ iff $\overline{\mathcal{S}}$ is PCE_f over $\mathcal{U}$. In particular, if $\mathcal{U}$ is defined by a theory V, then a sentence ϕ is PSC_f modulo V iff $\neg\phi$ is PCE_f modulo V.*

Theorem 5.4. *Let V and $T(\bar{x})$ be theories.*

1. *$T(\bar{x})$ is PCE_f modulo V iff $T(\bar{x})$ is equivalent modulo V to a theory of Π^0_2 formulae, all of whose free variables are among $\bar{x}$.*
2. *If $T(\bar{x})$ is PSC_f modulo V, then $T(\bar{x})$ is equivalent modulo V to a theory of Σ^0_2 formulae, all of whose free variables are among $\bar{x}$. However, the converse does not hold. There exist theories V and T such that (i) each sentence of T is a Σ^0_2 sentence having exactly one existential quantifier, and (ii) T is not PSC_f modulo V.*

Corollary 5.5. *Given a theory V, each of the following holds.*

1. *A formula $\phi(\bar{x})$ is PSC_f modulo V iff $\phi(\bar{x})$ is equivalent modulo V to a Σ^0_2 formula whose free variables are among $\bar{x}$.*
2. *A formula $\phi(\bar{x})$ is PCE_f modulo V iff $\phi(\bar{x})$ is equivalent modulo V to a Π^0_2 formula whose free variables are among $\bar{x}$.*

5.1. PSC_f (resp. PCE_f) vis-à-vis PSC (resp. PCE)

Corollaries 4.4, 4.6 and 5.5 yield the following result that is not obvious from the definitions of the properties concerned.

Corollary 5.6. *A sentence is PSC_f (resp. PCE_f) modulo a theory V iff it is PSC (resp. PCE) modulo V.*

Thus, given a sentence ϕ that is PSC_f (resp. PCE_f), there exists $k \in \mathbb{N}$ such that ϕ is $PSC(k)$ (resp. $PCE(k)$). This raises the question: is k computable? It turns out that, in general, k cannot be bounded by any recursive function of the length of ϕ. We present a short discussion on this in Appendix A.

While every *PCE* theory is trivially also a PCE_f theory, the following result, in contrast to Corollary 5.6, shows that PCE_f theories are, in general, strictly more expressive than *PCE* theories. We however do not know if PSC_f theories are more expressive than *PSC* theories.

Proposition 5.7. *There are theories V and T such that T is PCE_f modulo V but T is not PCE modulo V.*

Proof. Let V be the theory defining the class of all undirected graphs. Let T be a Π^0_1 theory over graphs asserting that there is no cycle of length k for any $k \in \mathbb{N}$. Then T defines the class $\mathcal{S}$ of all acyclic graphs, and is PCE_f modulo V by Theorem 5.4(1). Suppose T is PCE modulo V, whence T is $PCE(k)$ modulo V for some $k \in \mathbb{N}$. Then $\mathcal{S}$ is $PCE(k)$ modulo the class of models of V. By Lemma 3.9, $\overline{\mathcal{S}}$ (the complement of $\mathcal{S}$) is $PSC(k)$ modulo the class of models of V. Now consider a cycle G of length $k+1$. Clearly, G is in $\overline{\mathcal{S}}$ but every proper substructure of G is in $\mathcal{S}$. This contradicts our earlier inference that $\overline{\mathcal{S}}$ is $PSC(k)$ modulo the class of models of V. □

6. Comparisons with existing notions and results in literature

Corollaries 4.4 and 4.6 provide parameterized generalizations of the extensional and substructural forms respectively, of the Łoś–Tarski theorem for sentences over arbitrary finite vocabularies. These can therefore be collectively regarded as the *generalized Łoś–Tarski theorem for sentences*. This generalization can also be seen as providing new semantic characterizations of the Π^0_2 and Σ^0_2 prefix classes of FO sentences. Likewise, Theorems 4.1 and 5.4(1) provide new semantic characterizations of theories of $\forall^k\exists^*$ sentences and theories of Π^0_2 sentences respectively. There are other characterizations of the Π^0_2 and Σ^0_2 prefix classes in the literature via preservation under unions of ascending chains, preservation under intersections of descending chains and preservation under Keisler's 1-sandwiches [3]. However none of these characterizations relate the *count* of quantifiers to any model-theoretic properties. Our results therefore provide sharper semantic characterizations than those in the literature.

Furthermore, our notions of $PSC(k)$ and $PCE(k)$ are combinatorial and finitary in nature, and remain non-trivial over finite structures as well. This is in contrast to all of the aforementioned notions from the literature, that are trivially true for all sentences over any class of finite structures. This motivates the following question: Are there interesting classes of finite structures over which the generalized Łoś–Tarski theorem for sentences, holds? Recently, we answered this question affirmatively in [18]. Specifically, we identified a logic-based combinatorial property of classes of finite structures that entails an effective version of the generalized Łoś–Tarski theorem for sentences. We showed that this combinatorial property is enjoyed by various interesting classes of finite structures like words, trees (represented as partial orders), structures of bounded tree-depth, grids of bounded dimension, various classes of co-graphs like all co-graphs, complete graphs, complete n-partite graphs for each natural number n, threshold graphs, etc. The generalized Łoś–Tarski theorem then holds (in effective form) over all these classes.

In summary, the notions of $PSC(k)$ and $PCE(k)$ have not only enabled obtaining new and sharper preservation theorems in classical model theory, they have also been shown to be useful in obtaining new preservation theorems in the context of finite model theory.

7. Conclusion and future work

In this paper, we presented dual parameterized preservation properties that generalize the classical properties of preservation under substructures and preservation under extensions. We syntactically characterized sentences having these properties, obtaining as a consequence, a parameterized generalization of the Łoś–Tarski theorem for sentences. Our results provide semantic characterizations of the $\exists^k\forall^*$ and $\forall^k\exists^*$ prefix classes of FO sentences, for each natural number k, and are thus sharper than existing characterizations in the literature, of the Σ^0_2 and Π^0_2 prefix classes of FO sentences.

The following questions naturally arise from the current work, and are proposed as a part of future work.

1. We would like to investigate what syntactic subclasses of FO theories correspond exactly to $PSC(k)$ and PSC_f theories. As Theorems 4.5 and 5.4(2) show, these syntactic classes must semantically be subclasses of Σ^0_2 theories. For $PSC(k)$ theories, in addition to verifying whether Hypothesis 4.19 is true, we would further like to investigate what syntactic subclass of theories of $\exists^k\forall^*$ sentences characterizes $PSC(k)$ theories,

assuming Hypothesis 4.19 holds. A technique to identify the latter syntactic subclass is to examine the syntactic properties of the FO theories given by Proposition 4.23, and exploit the fact that these theories are obtained from the finite approximations of the infinitary sentences of $[\bigvee] \left[\exists^k \bigwedge\right] \Pi^0_1$.

2. As "converses" to the investigations above, and as analogues of the semantic characterizations of Π^0_2 theories and theories of $\forall^k\exists^*$ sentences by PCE_f and $PCE(k)$ respectively, we would like to semantically characterize Σ^0_2 theories and theories of $\exists^k\forall^*$ sentences, in terms of properties akin to (though not the same as) PSC_f and $PSC(k)$.

3. It is conceivable that many semantic properties of FO theories have natural and intuitive descriptions/characterizations in infinitary logics (Lemma 4.22 gives one such example). Then, results like Proposition 4.23 can be seen as "compilers" (in the sense of compilers used in computer science), in that they give a means of translating a "high level" description – via infinitary sentences that are known to be equivalent to FO theories – to an equivalent "low level" description – via FO theories. The latter FO theories are obtained from appropriately defined finite approximations of the infinitary sentences. It would therefore be useful to investigate other infinitary logics and their fragments for which such compiler-results can be established. An interesting logic to investigate in this regard would be $\mathcal{L}_{\omega_1,\omega}$, which is well-known to enjoy excellent model-theoretic properties despite compactness theorem not holding of it [11].

4. The results of this paper give characterizations of Σ^0_2 and Π^0_2 sentences in which the number of quantifiers in the leading block is given. As natural generalizations of these results, we can ask for characterizations of Σ^0_n and Π^0_n sentences for each $n \geq 2$, where the numbers of quantifiers in all the n blocks are given, and further extend these characterizations to theories. It may be noted that the results in the literature characterize Σ^0_n and Π^0_n theories as a whole and do not provide the finer characterizations suggested here.

5. The results of Section 5.1 have been used to obtain new proofs of known inexpressibility results in FO [16]. We would like to investigate more such applications of our results.

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Appendix A

Corollary 5.6 tells that given a sentence ϕ that is PSC_f (resp. PCE_f) modulo a theory V, there exists $k \in \mathbb{N}$ such that ϕ is $PSC(k)$ (resp. $PCE(k)$) modulo V. This raises the question: is k computable? The following proposition answers the aforesaid question in the negative. Below, a *relational* sentence is a sentence over a vocabulary that does not contain any function symbols. Let the length of a sentence ϕ be denoted by $|\phi|$.

Proposition A.1. *Let V be the empty theory. For every recursive function $\nu : \mathbb{N} \to \mathbb{N}$, we have the following:*

1. *There is a relational Π^0_2 sentence ϕ that is PSC_f modulo V but that is not $PSC(k)$ modulo V for any $k \leq \nu(|\phi|)$.*
2. *There is a relational Σ^0_2 sentence ϕ that is PCE_f modulo V but that is not $PCE(k)$ modulo V for any $k \leq \nu(|\phi|)$.*

Towards the proof of the above proposition, we first present a recent unpublished result of Rossman [13].

Theorem A.2 *(Rossman, 2012). Let V be the empty theory. For every recursive function $\nu : \mathbb{N} \to \mathbb{N}$, there exists a relational Σ_2^0 sentence ϕ that is PS modulo V, and for which every equivalent Π_1^0 sentence has length at least $\nu(|\phi|) + 1$.*

Theorem A.2 gives a non-recursive lower bound on the length of Π_1^0 sentences equivalent to sentences that are *PS* (in terms of the lengths of the latter sentences). This strengthens the non-elementary lower bound proved in [5].

Corollary A.3. *Let V be the empty theory. For every recursive function $\nu : \mathbb{N} \to \mathbb{N}$, there exists a relational Σ_2^0 sentence ϕ that is PS modulo V, and for which every equivalent Π_1^0 sentence has at least $\nu(|\phi|) + 1$ universal variables.*

Proof. We show below that there is a monotone recursive function $\rho : \mathbb{N} \to \mathbb{N}$ such that if ξ is a Π_1^0 sentence with n variables, then the shortest (in terms of length) Π_1^0 sentence equivalent to ξ has length at most $\rho(n)$. That would prove this corollary as follows. Suppose there is a recursive function $\nu : \mathbb{N} \to \mathbb{N}$ such that for each relational Σ_2^0 sentence ψ that is *PS* modulo V, there is an equivalent Π_1^0 sentence having at most $\nu(|\psi|)$ universal variables. Then consider the recursive function $\theta : \mathbb{N} \to \mathbb{N}$ given by $\theta(n) = \rho(\nu(n))$ and let ϕ be the relational Σ_2^0 sentence given by Theorem A.2 for the function θ. Then ϕ is *PS* modulo V and the shortest Π_1^0 sentence equivalent to ϕ has length $> \theta(|\phi|)$. By the assumption about ν above, there is a Π_1^0 sentence equivalent to ϕ having at most $\nu(|\phi|)$ universal variables. Whence there is a Π_1^0 sentence equivalent to ϕ whose length is at most $\rho(\nu|\phi|) = \theta(|\phi|)$ – a contradiction.

Let ξ be a universal sentence given by $\xi = \forall^n \bar{z}\beta(\bar{z})$. Let the vocabulary of ξ be τ and the maximum arity of any predicate of τ be q. Then the number k of atomic formulae of τ having variables from $\bar{z}$ is at most $|\tau| \cdot n^q$. It follows that the length r of the disjunctive normal form, say α, of β satisfies $r \leq (d \cdot k \cdot 2^k)$ for some constant $d \geq 1$. Then ξ is equivalent to the sentence $\gamma = \forall^n \bar{z}\alpha(\bar{z})$; the size of γ is at most $e \cdot (n + r)$ for some constant $e \geq 1$. Since k and r are bounded by monotone recursive functions of n, so is the length of γ. □

Using Corollary A.3, we can prove Proposition A.1 as follows.

Proof of Proposition A.1. We give the proof for Part 1. The negation of the sentence ϕ showing Part 1 proves Part 2. Also, we omit the mention of V for the sake of readability.

Suppose there is a recursive function $\nu : \mathbb{N} \to \mathbb{N}$ such that if ξ is a relational Π_2^0 sentence that is PSC_f, then ξ is $PSC(k)$ for some $k \leq \nu(|\xi|)$. In other words, for ξ as mentioned, every model of ξ has a crux of size at most $\nu(|\xi|)$. Consider the recursive function $\rho : \mathbb{N} \to \mathbb{N}$ given by $\rho(n) = \nu(n + 1)$. Then, for the function ρ, consider the relational Σ_2^0 sentence ϕ given by Corollary A.3. The sentence ϕ is *PS* and every Π_1^0 sentence equivalent to it has $> \rho(|\phi|)$ number of universal variables. Now the Π_2^0 sentence ψ given by $\psi = \neg\phi$ is equivalent to a Σ_1^0 sentence. Since Σ_1^0 sentences are *PSC*, and hence PSC_f, it follows that ψ is PSC_f. Now, by our assumption about ν above, every model of ψ has a crux of size at most $\nu(|\psi|) = \nu(|\phi| + 1) = \rho(|\phi|)$. Then all minimal models of ψ have size at most $\rho(|\phi|) + q$, where q is the number of constant symbols in the vocabulary of ϕ. Using the fact that ψ is preserved under extensions, it is easy to construct a Σ_1^0 sentence having $\rho(|\phi|)$ number of existential variables, that is equivalent to ψ. Whereby ϕ is equivalent to a Π_1^0 sentence having $\rho(|\phi|)$ number of universal variables – a contradiction. □

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