A Generalization of the Łoś-Tarski Preservation Theorem

Ph.D. defence

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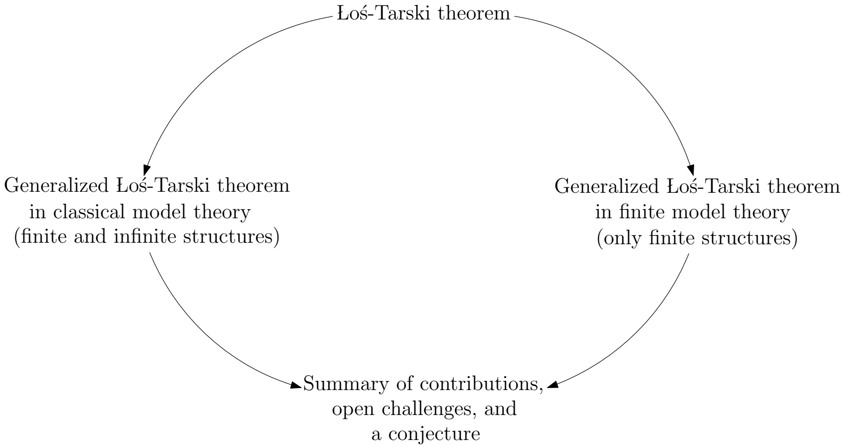
Introduction

- Classical model theory studies the relationship between the properties of definable classes of structures and the properties of their defining formulae.
- A preservation theorem characterizes (definable) classes of structures closed under a given model theoretic operation.
- Preservation under substructures – the Łoś-Tarski theorem.
- Connections of the Łoś-Tarski theorem with various mathematical disciplines
- Inspired the whole area of preservation theorems, and was amongst the earliest applications of Gödel's compactness theorem and the downward Löwenheim-Skolem theorem.

Introduction (Contd.)

- Finite model theory has similar aims as classical model theory but concerns itself with only finite structures.
- Unfortunately, most preservation theorems fail in the finite. This includes the Łoś-Tarski theorem.
- Recent research (by Atserias, Dawar, Grohe, Kolaitis) has focussed on "recovering" preservation theorems by considering classes of finite structures having good algorithmic properties.
- These include classes of bounded degree, those that are acyclic and more generally of bounded tree-width, and turn out to be "well-behaved" model-theoretically as well.
- Investigating such well-behavedness is an active and current line of research.

Talk outline



Some assumptions and notation for the talk

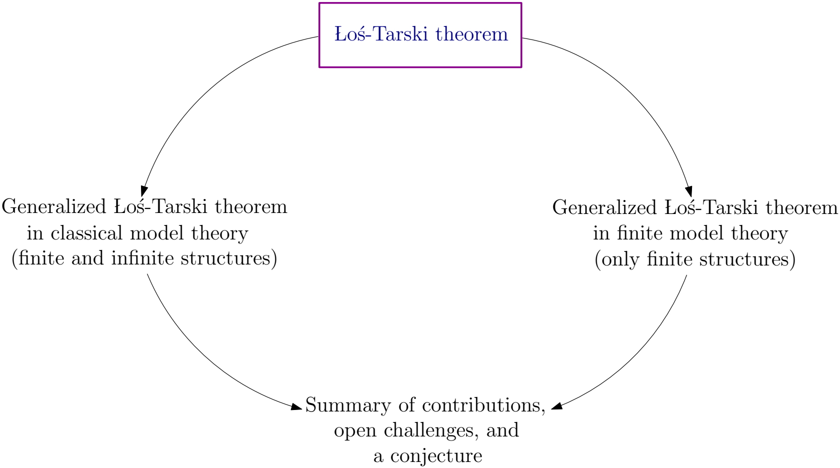
Assumptions:

- First order (FO) logic.
- Relational vocabularies (i.e. only predicates).

Notations:

- $\forall^* = \forall x_1 \ldots \forall x_n$(quantifier-free formula in $x_1, \ldots x_n$)
- $\exists^k\forall^* = \exists x_1 \ldots \exists x_k \forall y_1 \ldots \forall y_n$
  (quantifier-free formula in $x_1, \ldots, x_k, y_1, \ldots, y_n$)
- $\Sigma_2 = \bigcup_{k \geq 0} \exists^k\forall^*$
- Similarly, $\exists^*, \forall^k\exists^*$ and Π_2
- $\mathcal{A}_1 \subseteq \mathcal{A}_2$ means $\mathcal{A}_1$ is a substructure of $\mathcal{A}_2$. For graphs, $\subseteq$ means *induced subgraph*.
- $\mathsf{U}_{\mathcal{A}}$ = universe of $\mathcal{A}$.

Talk outline



Classical preservation properties

Definition

A sentence φ is said to be preserved under substructures, denoted φ is PS, if $\big((\mathcal{A} \models \varphi) \wedge (\mathcal{B} \subseteq \mathcal{A})\big) \rightarrow \mathcal{B} \models \varphi$.

- E.g.: $\varphi = \forall x \forall y E(x, y)$ describing the class of cliques, is PS.

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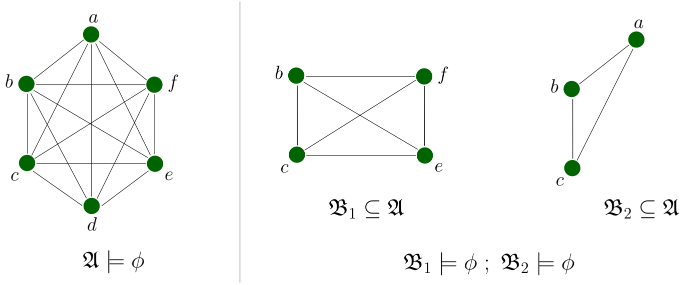

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- In general, every $\forall^*$ sentence is PS.

Definition

A sentence φ is said to be preserved under extensions, denoted φ is PE, if $\neg\varphi$ is PS.

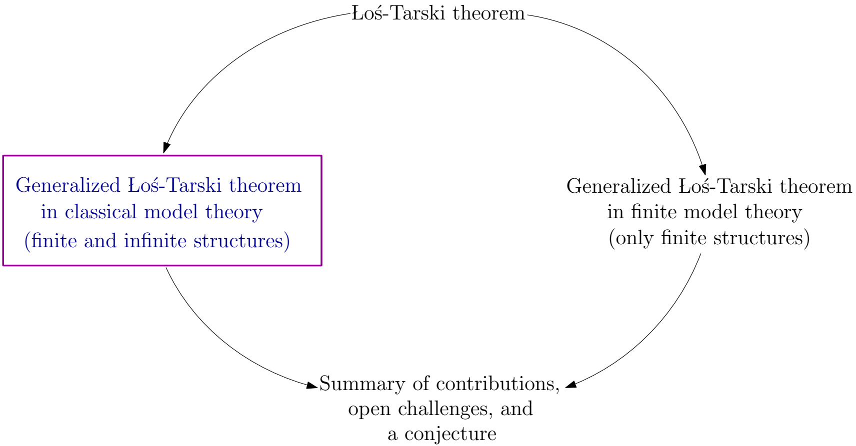
- PS and PE are dual properties. Every $\exists^*$ sentence is PE.

The Łoś-Tarski theorem: LT

Theorem (Łoś-Tarski, 1954-55)

1. A sentence is PS iff it is equivalent to a $\forall^*$ sentence.
2. A sentence is PE iff it is equivalent to an $\exists^*$ sentence.

Talk outline



New parameterized generalizations of the classical properties

Preservation under substructures modulo k-cruxes

Definition

A sentence φ is said to be preserved under substructures modulo k-cruxes, abbreviated φ is $PSC(k)$, if for each model $\mathcal{A}$ of φ, there is a subset C of $\mathsf{U}_{\mathcal{A}}$, of size $\leq k$, s.t.

$$\big((\mathcal{B} \subseteq \mathcal{A}) \wedge (C \subseteq \mathsf{U}_{\mathcal{B}})\big) \rightarrow \mathcal{B} \models \varphi$$

- The set C is called a k-crux of $\mathcal{A}$ (w.r.t. φ).
- Easy to see that

$$PS = PSC(0)$$
$$PSC(0) \subseteq PSC(1) \subseteq PSC(2) \subseteq \ldots$$

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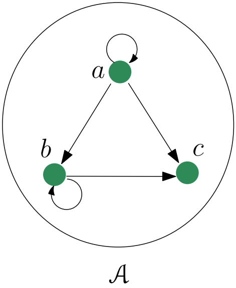

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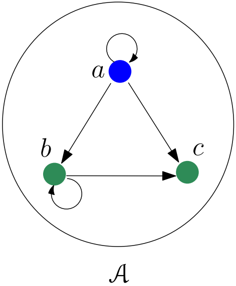

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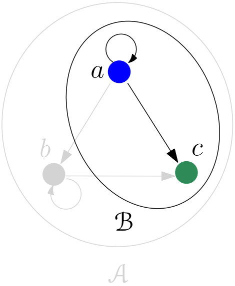

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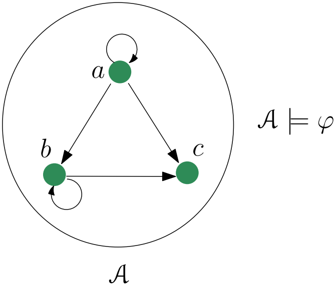

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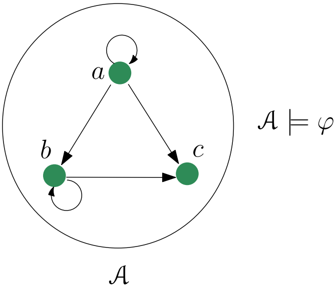

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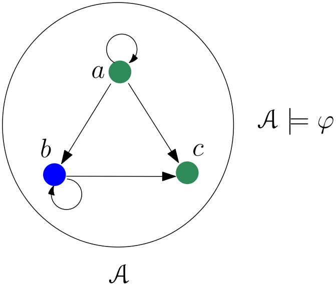

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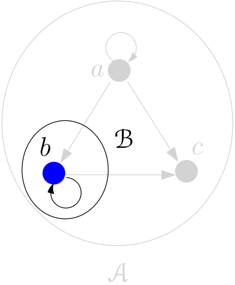

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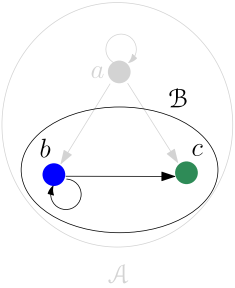

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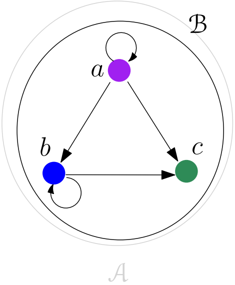

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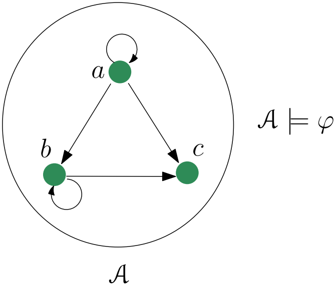

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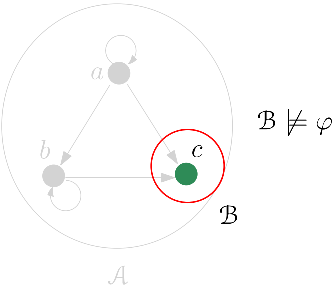

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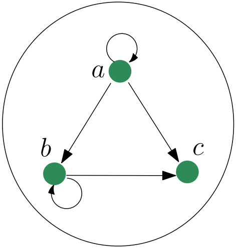


$\mathcal{A} \models \varphi$

$\mathcal{A}$

- Any witness for x is a 1-crux. Thus φ is $PSC(1)$.
- There can be 1-cruxes that are not witnesses for x.
- Observe that φ is not PS. Then $PS \subsetneq PSC(1)$.
- More generally, $PSC(0) \subsetneq PSC(1) \subsetneq PSC(2) \subsetneq \ldots$.

Natural properties of computer science interest are $PSC(k)$

1.	Bounded degree	$PSC(0)$
2.	G-freeness for any G (eg. triangle-freeness)	$PSC(0)$
3.	Bounded diameter (eg. cliqueness)	$PSC(0)$
4.	Vertex cover of size $\leq k$	$PSC(k)$
5.	Dominating set of size $\leq k$	$PSC(k)$
6.	Independent set of size $\geq k$, clique of size $\geq k$	$PSC(k)$
7.	Edge cover of size $\leq k$	$PSC(2k)$
8.	Matching of size $\geq k$	$PSC(2k)$

The dual of $PSC(k)$ and some quick observations

Definition

A sentence φ is said to be preserved under k-ary covered extensions, abbreviated φ is $PCE(k)$, if $\neg\varphi$ is $PSC(k)$.

- $PCE(0)$ is exactly PE.
- Any $\exists^k \forall^*$ sentence ϕ is $PSC(k)$.
- Whereby, any $\forall^k \exists^*$ sentence is $PCE(k)$.

Question: What about the converses of the last two statements?

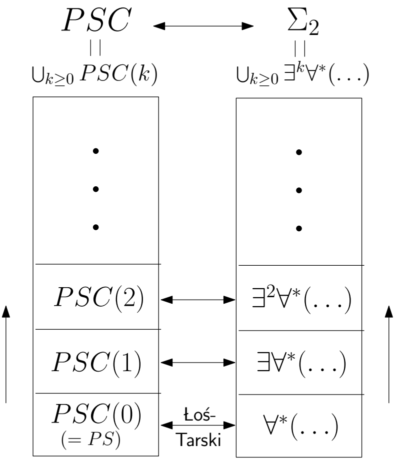
The generalized Łoś-Tarski theorem

The generalized Łoś-Tarski theorem: GLT(k)

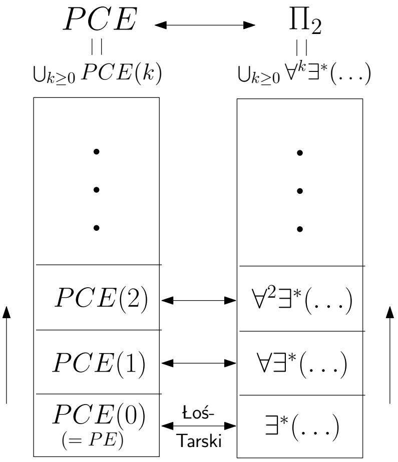
Theorem (GLT(k))

1. A sentence is $PSC(k)$ iff it is equivalent to an $\exists^k \forall^*$ sentence.
2. A sentence is $PCE(k)$ iff it is equivalent to a $\forall^k \exists^*$ sentence.

The generalized Łoś-Tarski theorem: GLT(k)



Substructural version



Extensional version

Extending GLT(k) to the case of theories

Extending earlier notions to theories

- A theory is a (possibly infinite) set of sentences, and can be understood as the conjunction of its sentences.
- The notions of $\forall^*, \exists^k\forall^*, \exists^*$ and $\forall^k\exists^*$ for sentences have natural extensions to theories.
- Likewise, the properties of $PS, PSC(k), PE$ and $PCE(k)$ have natural extensions to theories.

Question: What are characterizations of these extended properties?

LT for theories

Theorem (Łoś-Tarski, 1954-55)

1. A theory is PS iff it is equivalent to a $\forall^*$ theory.
2. A theory is PE iff it is equivalent to an $\exists^*$ theory.

Extending GLT(k) to theories

Theorem (GLT(k) for theories)

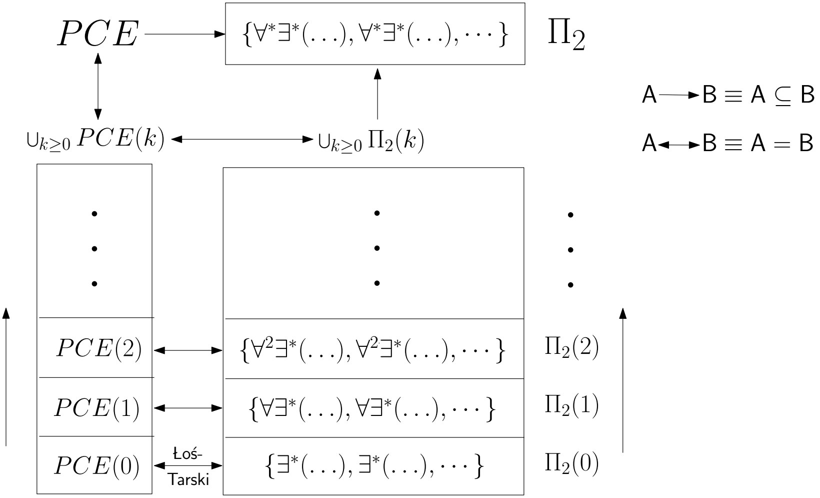
1. A theory is $PCE(k)$ iff it is equivalent to a $\forall^k \exists^*$ theory.
2. If a theory is $PSC(k)$, then it is equivalent to a Σ_2 theory.

Extending GLT(k) to theories

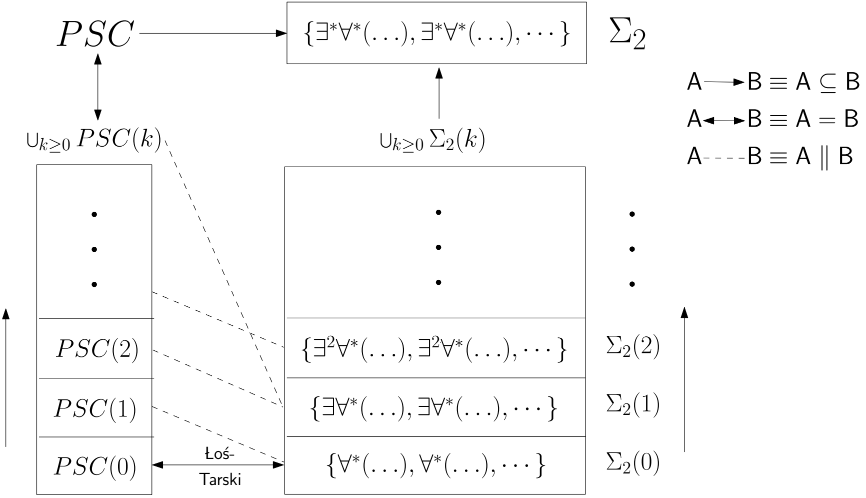
Theorem (GLT(k) for theories)

1. A theory is $PCE(k)$ iff it is equivalent to a $\forall^k \exists^*$ theory.
2. If a theory is $PSC(k)$, then it is equivalent to an $\exists^k \forall^*$ theory (under a well-motivated model-theoretic hypothesis).

Extensional version of GLT(k) for theories



Substructural version of GLT(k) for theories



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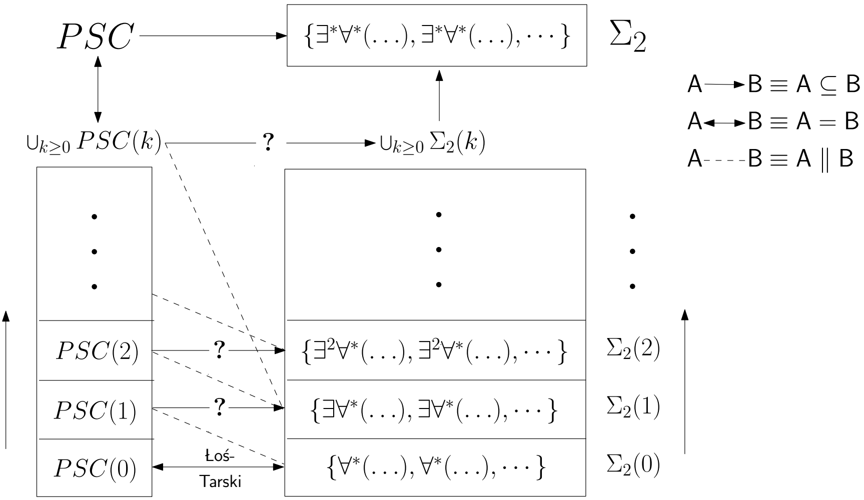

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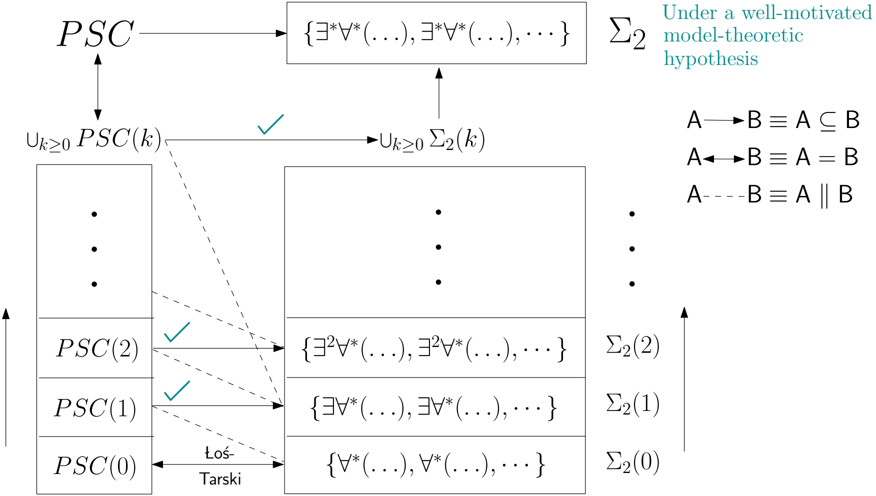
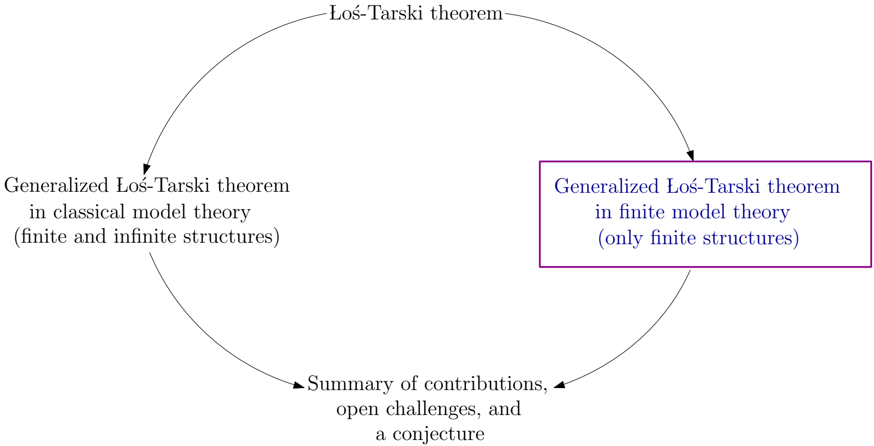


Techniques used

- Saturated structures
- Unions of ascending chains
- A new technique of “going above” FO and then “coming back”
  - Going above: Use an infinitary logic to express the property
  - Coming back: Use a “compiler result” to translate an infinitary sentence to an equivalent FO theory

Talk outline



Preservation properties 'over a given class of structures'

Definition

Let $\mathcal{U}$ be a given class of structures. A sentence φ is said to be preserved under substructures over $\mathcal{U}$, abbreviated φ is PS over $\mathcal{U}$, if for each structure $\mathcal{A}$ of $\mathcal{U}$, we have

$$\big((\mathcal{A} \models \varphi) \wedge (\mathcal{B} \subseteq \mathcal{A})$$

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- One can similarly define the preservation properties of $PE, PSC(k)$ and $PCE(k)$ over a class $\mathcal{U}$ of structures.
- One can then talk about preservation theorems *over* $\mathcal{U}$.
- All results seen so far have been over the class of all structures.

What happens in the finite?

LT in the finite

Proposition (Tait 1959, Gurevich-Shelah 1984)

LT fails over the class $\mathcal{U}$ of all finite structures. There is a sentence that is PS over $\mathcal{U}$ but is not equivalent, over $\mathcal{U}$, to any $\forall^*$ sentence.

Theorem (Atserias-Dawar-Grohe, 2008)

The LT holds over each of the following classes of graphs:

1. The class of all acyclic graphs.
2. The class of all graphs of degree $\leq d$, for each $d \in \mathbb{N}$.
3. The class of all graphs of tree-width $\leq d$, for each $d \in \mathbb{N}$.

GLT(k) in the finite

Proposition

GLT(k) fails over the class $\mathcal{U}$ of all finite structures, for each $k \geq 0$.

Proposition

GLT(k) fails over any hereditary class of finite graphs that has unbounded diameter, for each $k \geq 2$.

Summary: Over the classes identified by Atserias, Dawar and Grohe, LT holds but GLT(k) fails for each $k \geq 2$.

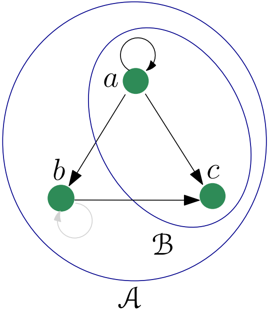
Investigating new classes of finite structures for GLT(k)

Can we identify structural properties (possibly abstract) of classes of finite structures, that are satisfied by interesting classes, and that admit GLT(k)?

Investigating new classes of finite structures for GLT(k)

Can we identify structural properties (possibly abstract) of classes of finite structures, that are satisfied by interesting classes, and that admit GLT(k)? And further, in effective form?

m-similarity of structures



$\mathcal{A}$ and $\mathcal{B}$ are 1-similar, but not 2-similar.

We say graphs $\mathcal{G}$ and $\mathcal{H}$ are m-similar, if $\mathcal{G}$ and $\mathcal{H}$ agree on all properties that can be expressed using FO sentences having quantifier nesting depth m.

A new logic based combinatorial property

The Equivalent Bounded Substructure Property

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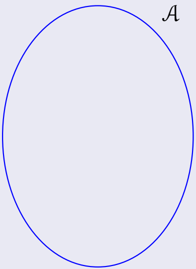

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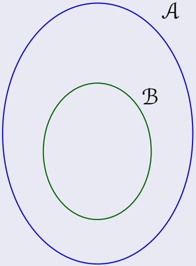

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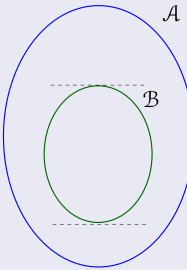

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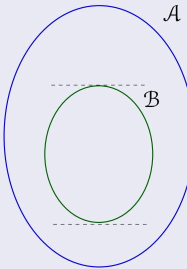

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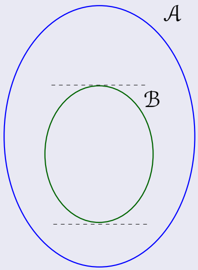

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"$\mathcal{A}$ has a small m-similar substructure"

The Equivalent Bounded Substructure Property

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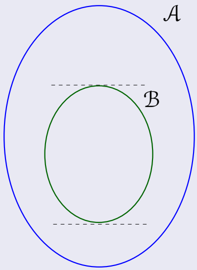
We say EBSP holds if there exists a witness function $\theta : \mathbb{N} \rightarrow \mathbb{N}$ such that



$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}| \leq \theta(m)$ and
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

"$\mathcal{A}$ has a small m-similar substructure"

The Equivalent Bounded Substructure Property

Definition

Given $k \in \mathbb{N}$, we say EBSP(k) holds if there is a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that

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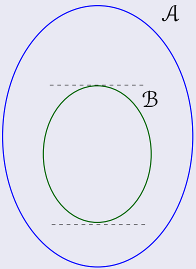

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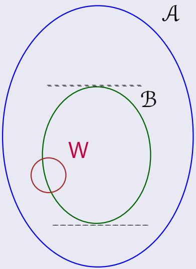
The Equivalent Bounded Substructure Property

Definition

Given $k \in \mathbb{N}$, we say EBSP(k) holds if there is a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that



$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\forall$ W $\subseteq$ U$_{\mathcal{A}}$ such that $|$W$| \leq k$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}| \leq \theta(m)$ and
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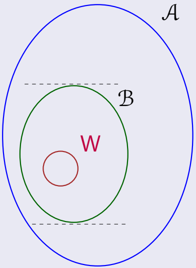

$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$

$\forall \mathsf{W} \subseteq \mathsf{U}_{\mathcal{A}}$ such that $|\mathsf{W}| \leq k$

$\exists \mathcal{B} \subseteq \mathcal{A}$

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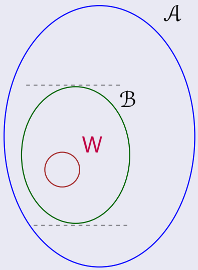

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" $\forall W \subseteq_k \mathcal{U}_{\mathcal{A}}$, $\mathcal{A}$ has a small m-similar substructure around W"

The Equivalent Bounded Substructure Property

Definition

Given a class $\mathcal{S}$ of structures and $k \in \mathbb{N}$, we say $\mathsf{EBSP}(\mathcal{S}, k)$ holds if there is a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that

The Equivalent Bounded Substructure Property

Definition

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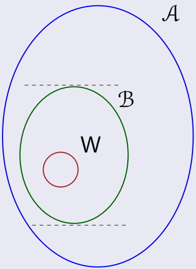

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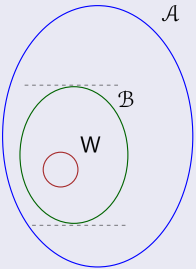
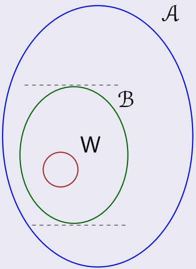


$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
$\forall$ W $\subseteq$ U$_{\mathcal{A}}$ such that $|\mathsf{W}| \leq k$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) $\mathcal{B} \in \mathcal{S}$ (ii) $\mathcal{B}$ contains W
(ii) $|\mathcal{B}| \leq \theta(m)$ and
(iii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

The Equivalent Bounded Substructure Property

Definition

Given a class $\mathcal{S}$ of structures and $k \in \mathbb{N}$, we say EBSP($\mathcal{S}, k$) holds if there is a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that

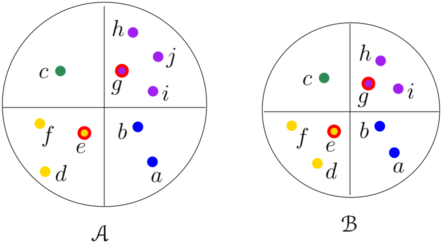


$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
$\forall$ W $\subseteq$ $U_{\mathcal{A}}$ such that $|W| \leq k$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) $\mathcal{B} \in \mathcal{S}$ (ii) $\mathcal{B}$ contains W
(ii) $|\mathcal{B}| \leq \theta(m)$ and
(iii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

" $\forall W \subseteq_k \mathcal{U}_{\mathcal{A}}$, $\mathcal{A}$ has a small m-similar substructure around W" – over $\mathcal{S}$

Example for EBSP($\mathcal{S}$, k)

$k = 2$
$m = 2$
$\mathsf{W} = \{b, i\}$

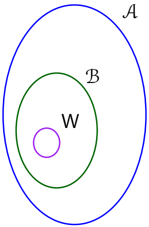


(i) $\mathcal{B} \subseteq \mathcal{A}$
(ii) $\mathsf{W} \subseteq \mathsf{U}_{\mathcal{B}}$
(iii) $|\mathcal{B}| \leq m \cdot 4 + k$
$= 10$
(iv) $\mathcal{B}$ is 2-similar to $\mathcal{A}$

- EBSP($\mathcal{S}$, k) holds with the witness function given by $\theta(m) = 4m + k$.

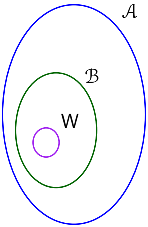
EBSP($\mathcal{S}, k$) – a finitary analogue of the downward Löwenheim-Skolem property

DLSP(κ)



$\forall \mathcal{A}$
$\forall$ W $\subseteq$ U$_{\mathcal{A}}$ such that $|\mathsf{W}| \leq \kappa$
$\exists \mathcal{B}$
(i) $\mathcal{B} \subseteq \mathcal{A}$ (ii) $\mathcal{B}$ contains W
(iii) $\mathcal{B}$ has size at most κ, and
(iv) $\mathcal{B}$ is FO-similar to $\mathcal{A}$

EBSP($\mathcal{S}, k$) for a fixed m



There is a natural number p such that

$\forall \mathcal{A} \in \mathcal{S}$
$\forall$ W $\subseteq$ U$_{\mathcal{A}}$ such that $|\mathsf{W}| \leq k$
$\exists \mathcal{B} \in \mathcal{S}$
(i) $\mathcal{B} \subseteq \mathcal{A}$ (ii) $\mathcal{B}$ contains W
(iii) $\mathcal{B}$ has size at most p, and
(iv) $\mathcal{B}$ is m-similar to $\mathcal{A}$

EBSP($\mathcal{S}, k$) and GLT(k)

Theorem

Let $\mathcal{S}$ be a class of finite structures and $k \in \mathbb{N}$ be such that EBSP($\mathcal{S}, k$) holds. Then the following are true:

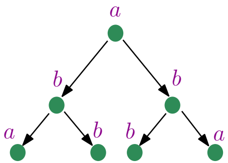
- GLT(k) holds over $\mathcal{S}$.
- If there is a computable witness function for EBSP($\mathcal{S}, k$), then there is an algorithm that translates a given $PSC(k)/PCE(k)$ sentence to an equivalent $\exists^k\forall^*/\forall^k\exists^*$ sentence.

Classes that satisfy EBSP$(\cdot, k)$

Posets satisfying EBSP$(\cdot, k)$

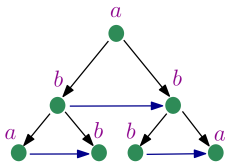
Words and trees (unordered, ordered, ranked)

- Classically studied structures



Unordered Σ-tree

$\Sigma = \{a, b\}$



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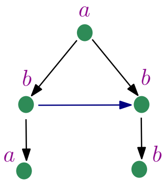
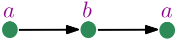


Ordered Σ-tree ranked by ρ

$\Sigma = \{a, b\}$; $\rho = \{a \to 2, b \to 1)\}$



Σ-word

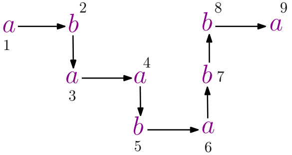
$\Sigma = \{0, 1\}$

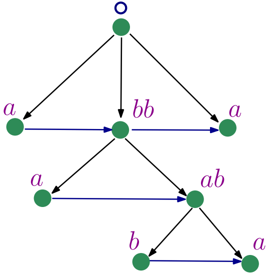
Nested words

- Introduced by Alur and Madhusudan in 2004 as joint generalization of words and ordered unranked trees.

$W = (abaababba, \rightsquigarrow)$

$\rightsquigarrow = \{(2,8),(4,7)\}$





Regular languages of words, trees and nested words

A regular language of words/trees/nested words is a class of words/trees/nested words that can be recognized by a finite word/tree/nested word automaton.

Theorem

Let $\mathcal{S}$ be a regular language of words, trees (unordered, ordered or ranked) or nested words. For each k, EBSP$(\mathcal{S}, k)$ holds with a computable witness function.

Graphs satisfying EBSP$(\cdot, k)$

m-partite cographs

- Hliněný, Nešetřil, et al. introduced in 2012, the class of m-partite cographs.

m-partite cographs

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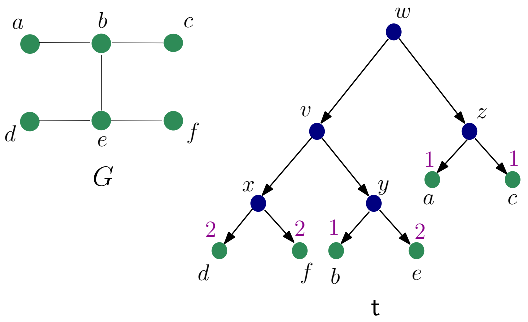
- An m-partite cograph G is a graph that has an m-partite cotree representation t:



Label set $= \{1, 2\}$
$f_x = f_z = 0$
$f_y = 1$
$f_v(2, 2) = 1$, else 0
$f_w(1, 1) = 1$, else 0

Important subclasses of m-partite cographs

- Cographs (1-partite cographs): complete graphs, complete k-partite graphs, threshold graphs, etc.
- Bounded tree-depth graphs
- Bounded shrub-depth graphs

All of the above classes are of active current interest for their excellent algorithmic and logical properties!

m-partite cographs and its subclasses satisfy EBSP$(\cdot, k)$

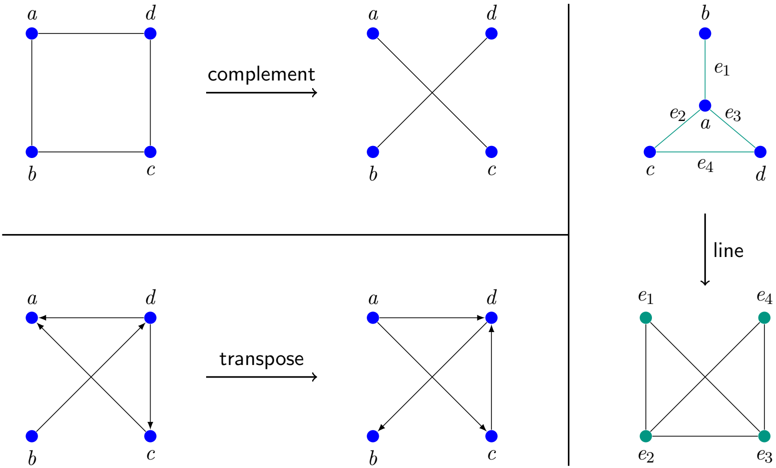
Theorem

Let $\mathcal{S}$ be a hereditary subclass of any of the following graph classes. For each k, EBSP$(\mathcal{S}, k)$ holds with a computable witness function.

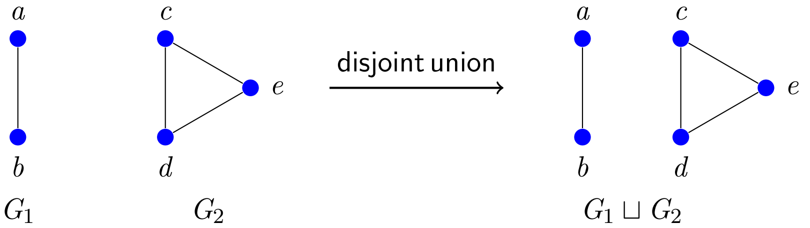
1. the class of m-partite cographs
2. any graph class of bounded shrub-depth
3. any graph class of bounded tree-depth
4. the class of cographs

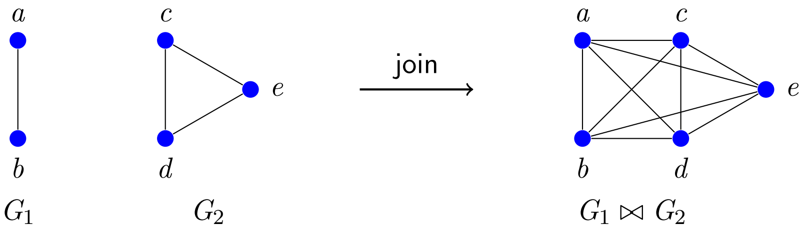
Constructing new classes satisfying EBSP$(\cdot, k)$

Unary operations on structures



Binary operations on structures





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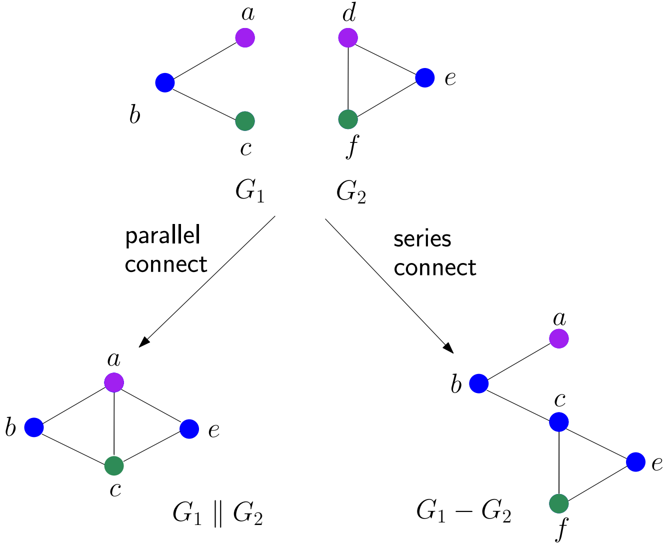

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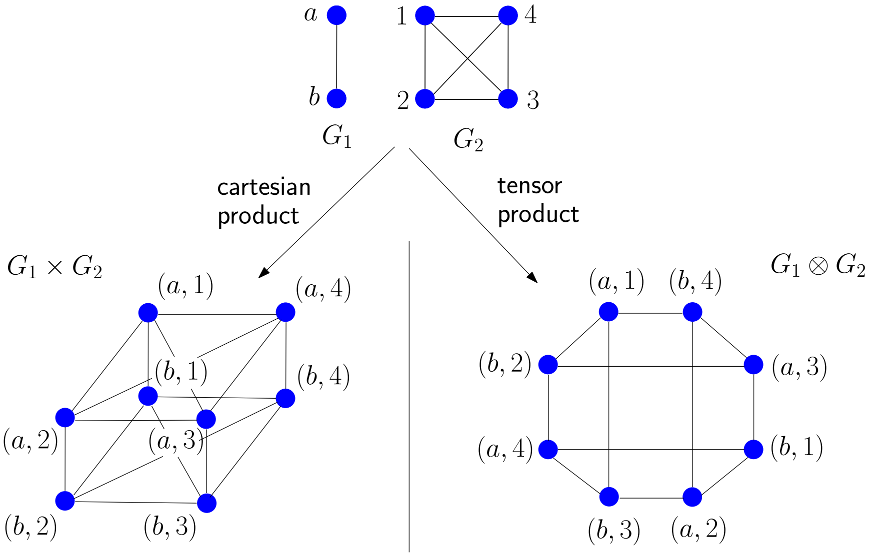
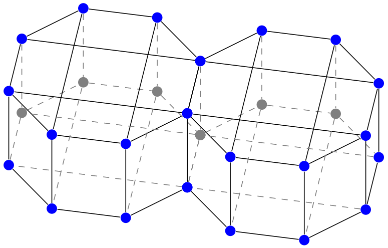
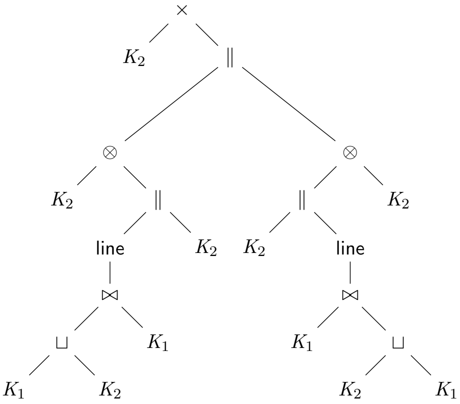


Generating graphs using trees of operations





K_1 = single vertex; K_2 = single edge

Closure of EBSP$(\cdot, \cdot)$ under unary operations

Theorem

Given a class $\mathcal{S}$, let $\mathcal{Z}$ be any one of the following classes.

1. Complement$(\mathcal{S})$
2. Transpose$(\mathcal{S})$
3. Line$(\mathcal{S})$

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- EBSP$(\mathcal{S}, k) \rightarrow$ EBSP$(\mathcal{Z}, k)$.

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- If EBSP$(\mathcal{S}, k)$ holds with a computable witness function, then so does EBSP$(\mathcal{Z}, k)$.

Closure of EBSP$(\cdot, \cdot)$ under binary operations

Theorem

Given classes $\mathcal{S}_1$ and $\mathcal{S}_2$, let $\mathcal{Z}$ be any one of the following classes.

1. Disjoint-union$(\mathcal{S}_1, \mathcal{S}_2)$
2. Join$(\mathcal{S}_1, \mathcal{S}_2)$
3. Series-connect$(\mathcal{S}_1, \mathcal{S}_2)$
4. Parallel-connect$(\mathcal{S}_1, \mathcal{S}_2)$
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The following are true:

- $\big(\text{EBSP}(\mathcal{S}_1, k) \wedge \text{EBSP}(\mathcal{S}_2, k)\big) \rightarrow \text{EBSP}(\mathcal{Z}, k)$ if $\mathcal{Z}$ in 1-4.
- $\big(\text{EBSP}(\mathcal{S}_1, 2k) \wedge \text{EBSP}(\mathcal{S}_2, 2k)\big) \rightarrow \text{EBSP}(\mathcal{Z}, k)$ if $\mathcal{Z}$ in 5-6.

Closure of EBSP$(\cdot, \cdot)$ under binary operations

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- $\big(\text{EBSP}(\mathcal{S}_1, 2k) \wedge \text{EBSP}(\mathcal{S}_2, 2k)\big) \rightarrow \text{EBSP}(\mathcal{Z}, k)$ if $\mathcal{Z}$ in 5-6.

Further, if the conjuncts in the antecedent have computable witness functions, then so does the consequent.

Techniques used to prove EBSP$(\cdot, k)$ for a class

- Key observation: Each of the structures $\mathcal{A}$ seen so far has a tree representation $\mathsf{t}_{\mathcal{A}}$.
- We perform "prunings" and "graftings" in $\mathsf{t}_{\mathcal{A}}$ that preserve the substructure and m-similarity relation between the newly formed subtree and $\mathsf{t}_{\mathcal{A}}$.
- We eventually get a small subtree of $\mathsf{t}_{\mathcal{A}}$ representing a small m-similar substructure of $\mathcal{A}$.
- Key technical features making the method work:
  - Finite number of different "m-similarity types"
  - Composition properties
- The techniques above have been incorporated into a single abstract theorem concerning trees.

Motivating question revisited

Can we identify structural properties (possibly abstract) of classes of finite structures, that are satisfied by interesting classes, and that admit $\mathsf{GLT}(k)$?

Motivating question revisited

Can we identify structural properties (possibly abstract) of classes of finite structures, that are satisfied by interesting classes, and that admit GLT(k)? And further, in effective form?

An answer to the motivating question

All these classes satisfy
GLT(k) in effective form for all k!

Posets

1. Words
2. Trees
   a. Unordered
   b. Ordered
   c. Ranked
3. Nested words

Graphs

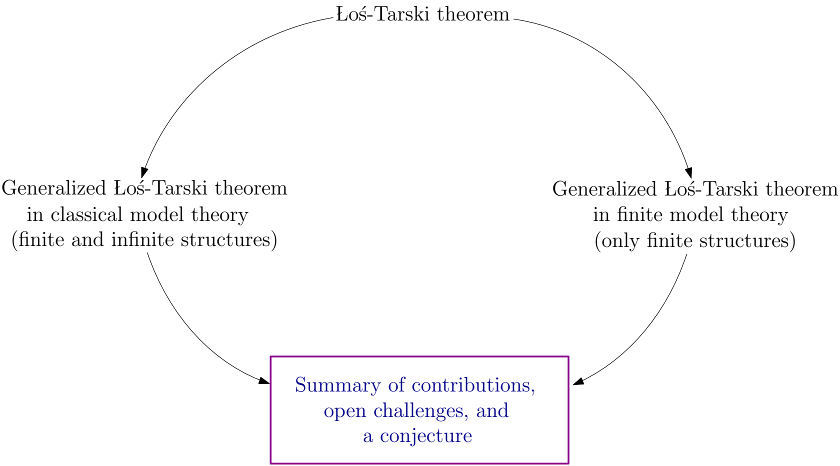
1. Cographs
2. Bounded tree-depth graphs
3. Bounded shrub-depth graphs
4. m-partite cographs

Classes generated using

1. Unary operations
   a. complement
   b. transpose
   c. line
2. Binary operations
   a. disjoint union
   b. join
   c. series-connect
   d. parallel-connect
   e. cartesian product
   f. tensor product

EBSP

Talk outline



Contributions to classical model theory

- Notions: $PSC(k)$ and $PCE(k)$
  - Admit natural variants that capture prenex FO sentences with n quantifier blocks
  - Are finitary and combinatorial in nature, and stay non-trivial over finite structures
- Results: GLT(k)
  - provides new and finer characterizations of Σ_2 and Π_2
  - relates counts of quantifiers to model-theoretic properties
  - can contribute to a keener understanding of the inner structure of model theory (extending Wilfrid Hodges' observation about the role of preservation theorems in model theory)
- Techniques: A new technique of syntactically describing a property in FO, by "going above" FO and then "coming back".

Contributions to finite model theory

- Notions: EBSP$(\mathcal{S}, k)$
  - Strong connections to classical model theory
  - Strong connections to computer science
  - Admits several natural variants
- Results:
  - Strengthening the result showing the failure of Łoś-Tarski theorem in the finite
  - A preservation theorem (GLT(k)) that enforces structural conditions
  - Characterizing prenex FO sentences with two quantifier blocks
  - Identifying a wide spectrum of classes of finite structures that are "well-behaved" model-theoretically
  - Relating the property of well-quasi-ordering with logic
- Techniques: An abstract theorem concerning tree representations

Open questions

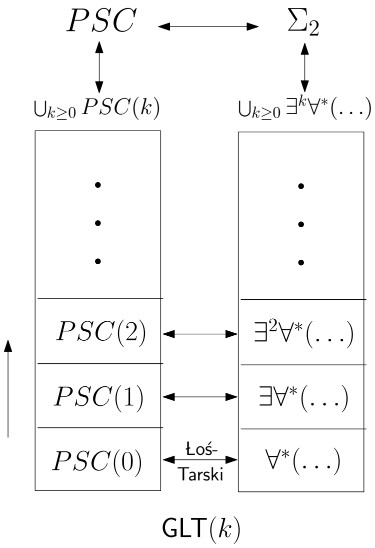
A. Classical model theory

- Getting an unconditional characterization of $PSC(k)$ theories

B. Finite model theory (questions concerning EBSP$(\mathcal{S}, k)$)

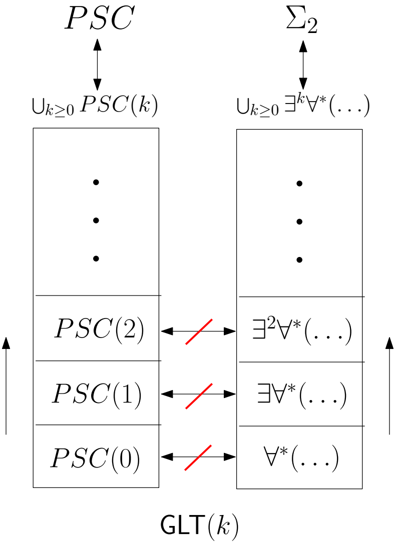
- [Model-theoretic] Other theorems of classical model theory that are entailed by EBSP$(\mathcal{S}, k)$ (Lyndon's positivity theorem, Craig's interpolation theorem, etc.)
- [Graph-theoretic] Structural characterization of EBSP$(\cdot, k)$. If not in general, under reasonable assumptions?
- [Probabilistic] Classes that satisfy the EBSP condition "with high probability".

A conjecture



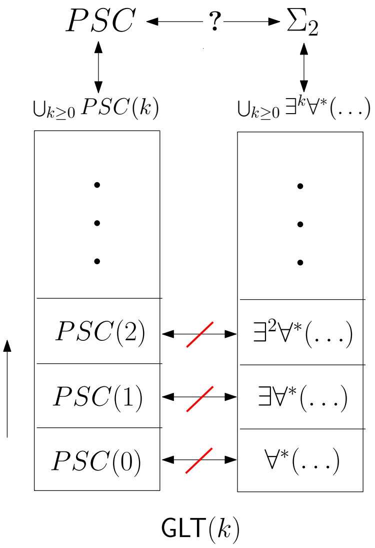
Over all structures
(finite and infinite)

A conjecture



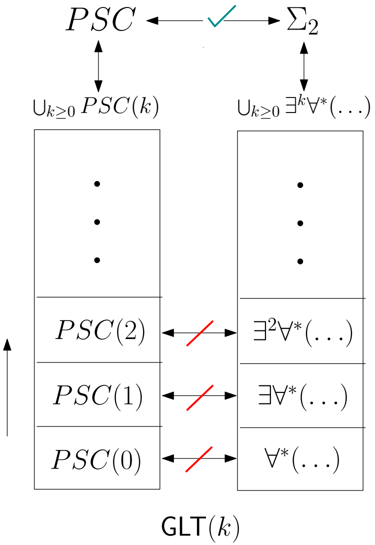
Over all finite structures

A conjecture



Over all finite structures

A conjecture



Over all finite structures

Conjecture:

$$PSC = \Sigma_2$$

Dhanyavād!

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