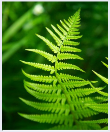
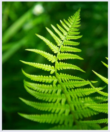


A Finitary Analogue of the Downward Löwenheim-Skolem Property



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Stockholm, Sweden

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Introduction

- The Downward Löwenheim-Skolem theorem (DLS) is amongst the earliest results in classical model theory.
- The first version of DLS is by Löwenheim in his paper Über Möglichkeiten im Relativkalkül (1915) and reads as follows:

  If a first order sentence over a countable vocabulary has an infinite model, then it has a countable model.
- Historically,

1915:	First version of DLS by Löwenheim
1920s:	Self-contained proof of Löwenheim's statement and various generalizations by Skolem
1936:	The most general version of DLS by Mal'tsev

- DLS + compactness = first order logic (Lindström, 1969).

Downward Löwenheim-Skolem theorem in the finite

- Does not make sense when taken as is.
- No recursive version of Löwenheim's statement:
  For every recursive function $f : \mathbb{N} \to \mathbb{N}$, there is an FO sentence φ such that φ has no model of size $< f(|\varphi|)$.
- Grohe showed a stronger negative result:
  For every recursive function $f : \mathbb{N} \to \mathbb{N}$, there is an FO sentence φ and $n \geq f(|\varphi|)$, such that φ has a model of each size $\geq n$ but no model of size $< n$.
- Quoting Grohe, the above counterexample "refutes almost all possible extensions of the classical Löwenheim-Skolem theorem to finite structures".

Classical theorems over classes of finite structures

- Most theorems from classical model theory fail over all finite structures (DLS, preservation theorems, interpolation theorems, etc.)
- Active research in last 15 years to "recover" classical theorems over classes interesting from structural and algorithmic perspectives.
- Acyclic, bounded degree, wide, bounded tree-width – Łoś-Tarski pres. theorem
- In addition to the above, quasi-wide classes, classes excluding atleast one minor – homomorphism pres. theorem
- No such studies in the literature for the DLS theorem.

Outline of the talk

A. Notions:

- The Downward Löwenheim-Skolem Property: DLSP
- The Equivalent Bounded Substructure Property: EBSP
- EBSP as a finitary analogue of DLSP

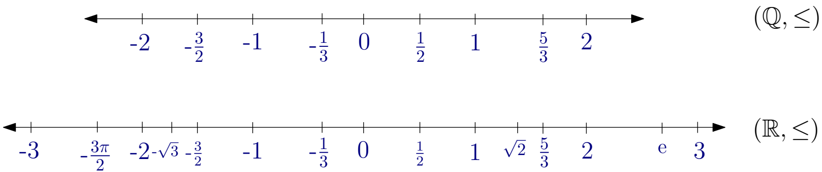
B. Results:

- Classes of finite structures satisfying EBSP
- Closure properties of EBSP
- Techniques and f.p.t. algorithms
- Connection with fractals

A. Notions

The Downward Löwenheim-Skolem Property

FO-similarity of structures



$(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are FO-similar

We say structures $\mathcal{A}$ and $\mathcal{B}$ are FO-similar, denoted $\mathcal{A} \equiv \mathcal{B}$, if $\mathcal{A}$ and $\mathcal{B}$ agree on all properties that can be expressed in FO.

The Downward Löwenheim-Skolem Property

Definition

We say DLSP holds if

The Downward Löwenheim-Skolem Property

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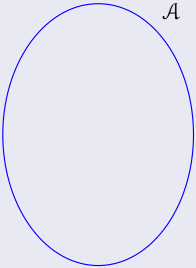
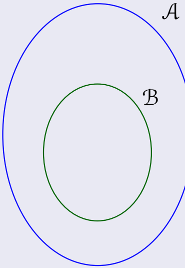


$\forall \mathcal{A}$

The Downward Löwenheim-Skolem Property

Definition

We say DLSP holds if



$\forall\, \mathcal{A}$
$\exists\, \mathcal{B} \subseteq \mathcal{A}$

The Downward Löwenheim-Skolem Property

Definition

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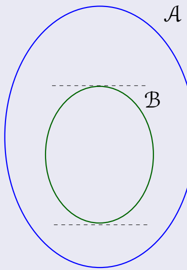

$\forall\, \mathcal{A}$
$\exists\, \mathcal{B} \subseteq \mathcal{A}$
(i) the size of $\mathcal{B}$ is $\leq \omega$

The Downward Löwenheim-Skolem Property

Definition

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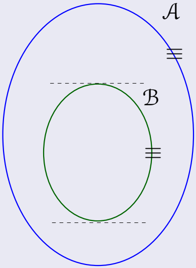
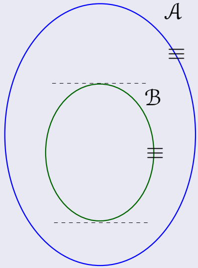


$\forall \mathcal{A}$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) the size of $\mathcal{B}$ is $\leq \omega$
(ii) $\mathcal{B}$ is FO-similar to $\mathcal{A}$

The Downward Löwenheim-Skolem Property

Definition

We say DLSP holds if



$\forall\, \mathcal{A}$

$\exists\, \mathcal{B} \subseteq \mathcal{A}$

(i) the size of $\mathcal{B}$ is $\leq \omega$

(ii) $\mathcal{B}$ is FO-similar to $\mathcal{A}$

"$\mathcal{A}$ has an FO-similar substructure of size $\leq \omega$"

The Downward Löwenheim-Skolem theorem

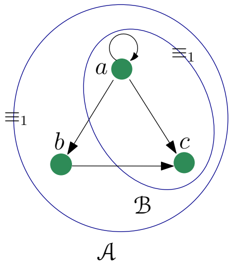
Theorem (Löwenheim 1915, Skolem 1920s)

DLSP holds over all infinite structures.

Adapting DLSP to the finite

m-similarity of structures

- In the finite, FO-similarity = isomorphism.
- Define similarity in terms of FO$[m]$ sentences, namely FO sentences of rank (quantifier nesting depth) at most m.
- We say $\mathcal{A}$ and $\mathcal{B}$ are m-similar, denoted $\mathcal{A} \equiv_m \mathcal{B}$, if $\mathcal{A}$ and $\mathcal{B}$ agree on all properties expressible using FO$[m]$ sentences.



$\mathcal{A}$ and $\mathcal{B}$ are 1-similar, but not 2-similar.

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if

The Equivalent Bounded Substructure Property

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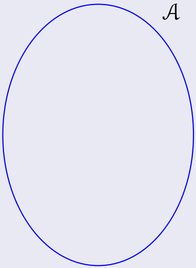

$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$

The Equivalent Bounded Substructure Property

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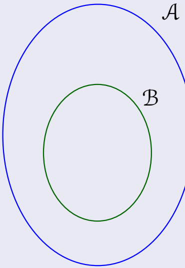

$\forall\, \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists\, \mathcal{B} \subseteq \mathcal{A}$

The Equivalent Bounded Substructure Property

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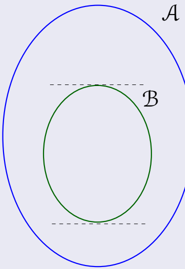

$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}|$ is bounded in m

The Equivalent Bounded Substructure Property

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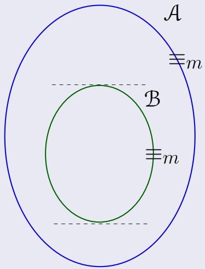


$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$

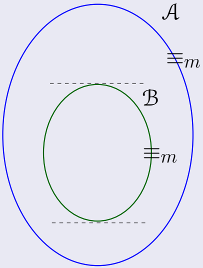
(i) $|\mathcal{B}|$ is bounded in m

(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if



$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$

(i) $|\mathcal{B}|$ is bounded in m

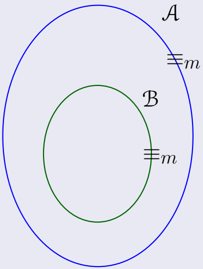
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

"$\mathcal{A}$ has a small m-similar substructure"

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if there exists a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that



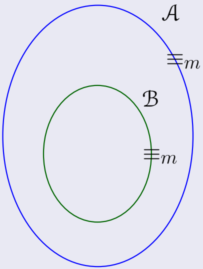
$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}| \leq \theta(m)$
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

"$\mathcal{A}$ has a small m-similar substructure"

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if there exists a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that



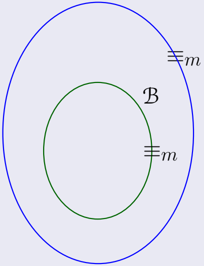
$\forall\, \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists\, \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}| \leq \theta(m)$
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

"$\mathcal{A}$ has a small m-similar substructure"

The Equivalent Bounded Substructure Property

Definition

Given a class $\mathcal{S}$ of finite structures, we say EBSP($\mathcal{S}$) holds if there is a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that



$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$

$\exists \mathcal{B} \subseteq \mathcal{A}, \quad \mathcal{B} \in \mathcal{S}$

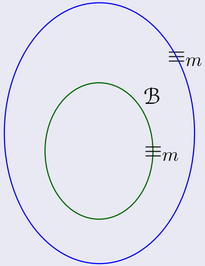
(i) $|\mathcal{B}| \leq \theta(m)$

(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

The Equivalent Bounded Substructure Property

Definition

Given a class $\mathcal{S}$ of finite structures, we say EBSP($\mathcal{S}$) holds if there is a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that

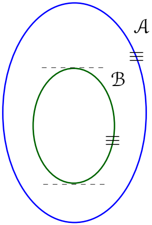


$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}, \quad \mathcal{B} \in \mathcal{S}$
(i) $|\mathcal{B}| \leq \theta(m)$
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

"$\mathcal{A}$ has a small m-similar substructure" – over $\mathcal{S}$

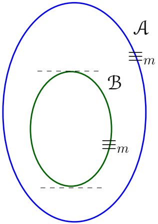
EBSP($\mathcal{S}$) as a finitary analogue of DLSP

DLSP



$\forall\, \mathcal{A}$
$\exists\, \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}| \leq \omega$
(ii) $\mathcal{B}$ is FO-similar to $\mathcal{A}$

EBSP($\mathcal{S}$) for a fixed m



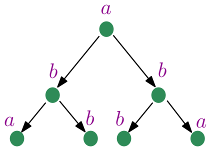
Let $p = \theta(m)$

$\forall\, \mathcal{A} \in \mathcal{S}$
$\exists\, \mathcal{B} \subseteq \mathcal{A}, \quad \mathcal{B} \in \mathcal{S}$
(i) $|\mathcal{B}| \leq p$
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

B. Results

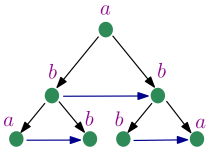
Classes that satisfy EBSP

Words, trees and nested words



Unordered Σ-tree

$\Sigma = \{a, b\}$



Ordered Σ-tree

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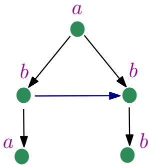
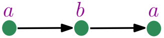


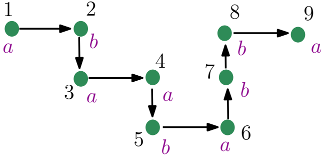
Ordered Σ-tree ranked by ρ

$\Sigma = \{a, b\}$; $\rho = \{a \to 2\,,\, b \to 1)\}$



Σ-word $w = aba$

$\Sigma = \{a, b\}$



Nested Σ-word $(w, \rightsquigarrow)$

$w = abaababba$

$\rightsquigarrow = \{(2, 8), (4, 7)\}$

$\Sigma = \{a, b\}$

Regular languages of words, trees and nested words

- A regular language of words/trees/nested words is a class of words/trees/nested words that can be recognized by a finite word/tree/nested word automaton.
- Recall: EBSP($\mathcal{S}$) says for each m, that a large $\mathcal{S}$-structure contains a small m-similar $\mathcal{S}$-substructure.

Theorem

Let $\mathcal{S}$ be a regular language of words, trees (unordered, ordered or ranked) or nested words. Then EBSP($\mathcal{S}$) holds with a computable witness function (which is non-elementary, in general).

m-partite cographs

- Hliněný, Nešetřil, et al. introduced in 2012, the class of m-partite cographs.
- This class is a special class of bounded clique-width graphs, and generalizes a number of important graph classes:
  - Cographs (1-partite cographs): complete graphs, complete k-partite graphs, threshold graphs, Turan graphs, etc.
  - Bounded tree-depth graphs
  - Bounded shrub-depth graphs
- All of the above classes are of active current interest for their excellent algorithmic and logical properties.

m-partite cographs and its subclasses satisfy EBSP

Theorem

Let $\mathcal{S}$ be a hereditary subclass of any of the following graph classes. Then EBSP($\mathcal{S}$) holds with a computable witness function. For classes with bounded parameters as below, there exist elementary witness functions.

1. the class of m-partite cographs
2. any graph class of bounded shrub-depth
3. any graph class of bounded tree-depth
4. the class of cographs

Well-quasi-ordering and EBSP

Definition

A class $\mathcal{S}$ of structures is said to be w.q.o. under embedding if for every infinite set $\{\mathcal{A}_1, \mathcal{A}_2, \ldots\}$ of structures of $\mathcal{S}$, there exist i, j such that $\mathcal{A}_i$ is embeddable in $\mathcal{A}_j$.

Theorem

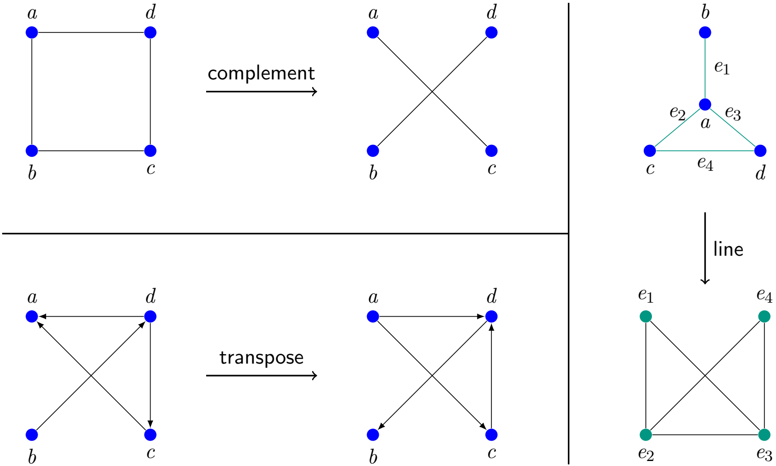
Let $\mathcal{S}$ be w.q.o. under embedding. Then EBSP$(\mathcal{S})$ is true (with uncomputable witness functions in general).

Applications: The following classes satisfy EBSP$(\mathcal{S})$:

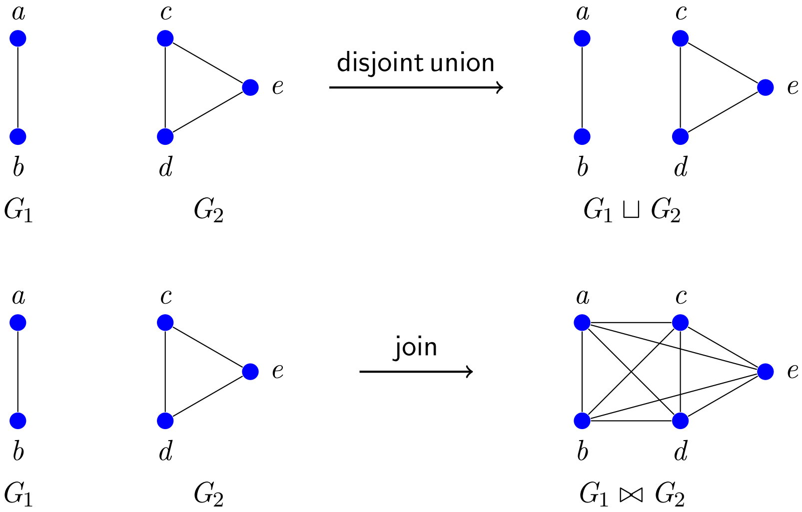
- k-letter graphs for each k (e.g. threshold graphs, unbounded interval graphs)
- k-uniform graphs for each k

Constructing new classes satisfying EBSP

Operations on structures



Operations on structures



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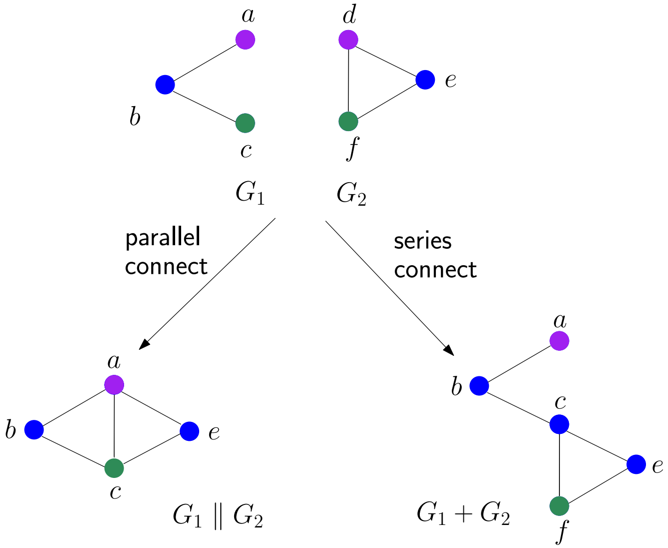
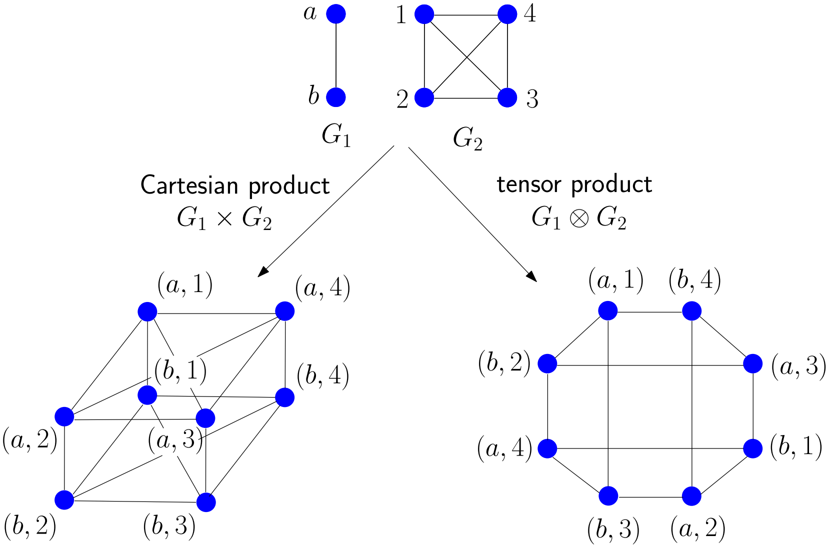
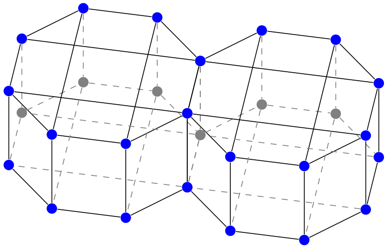
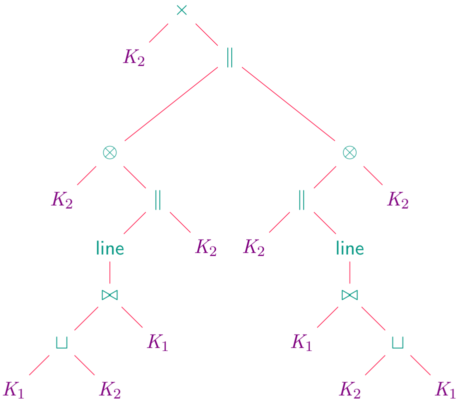


Operations on structures



Generating graphs using trees of operations





K_1 = single vertex; K_2 = single edge

Closure of EBSP under operations on structures

Theorem

Given a class $\mathcal{S}$, let $\mathcal{Z}$ be any one of the following classes.

1. Complement($\mathcal{S}$)
2. Transpose($\mathcal{S}$)
3. Line($\mathcal{S}$)

Then the following are true:

- EBSP($\mathcal{S}$) $\rightarrow$ EBSP($\mathcal{Z}$)
- If EBSP($\mathcal{S}$) holds with a computable/elementary witness function, then so does EBSP($\mathcal{Z}$).

Closure of EBSP under operations on structures

Theorem

Given classes $\mathcal{S}_1$ and $\mathcal{S}_2$, let $\mathcal{Z}$ be any one of the following classes.

1. Disjoint-union$(\mathcal{S}_1, \mathcal{S}_2)$
2. Join$(\mathcal{S}_1, \mathcal{S}_2)$
3. Series-connect$(\mathcal{S}_1, \mathcal{S}_2)$
4. Parallel-connect$(\mathcal{S}_1, \mathcal{S}_2)$
5. Cartesian-product$(\mathcal{S}_1, \mathcal{S}_2)$
6. Tensor-product$(\mathcal{S}_1, \mathcal{S}_2)$

Then the following are true:

- $\big(\mathsf{EBSP}(\mathcal{S}_1) \wedge \mathsf{EBSP}(\mathcal{S}_2)\big) \rightarrow \mathsf{EBSP}(\mathcal{Z})$
- If the conjuncts in the antecedent hold with computable/ elementary witness functions, then so does the consequent.

An overview of classes satisfying EBSP

Posets

1. Words
2. Trees
   a. Unordered
   b. Ordered
   c. Ranked
3. Nested words

Graphs

1. Cographs
2. Bounded tree-depth graphs
3. Bounded shrub-depth graphs
4. m-partite cographs
5. Graphs w.q.o. under embedding

Classes generated using

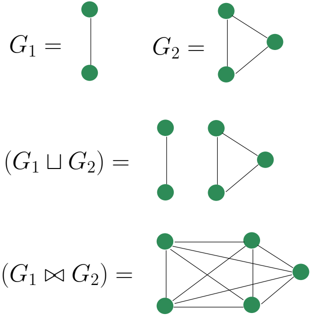
1. Unary operations
   a. complement
   b. transpose
   c. line
2. Binary operations
   a. disjoint union
   b. join
   c. series-connect
   d. parallel-connect
   e. Cartesian product
   f. tensor product

EBSP

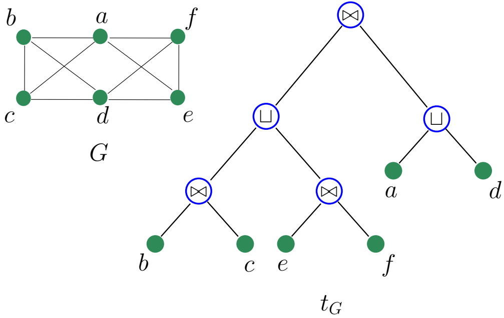
Techniques and f.p.t. algorithms

Illustrative example: Cographs

Generated from point graphs using disjoint union and join.



disjoint union $= \sqcup$; join $= \bowtie$



Cograph G and its cotree t_G

Some model-theoretic facts

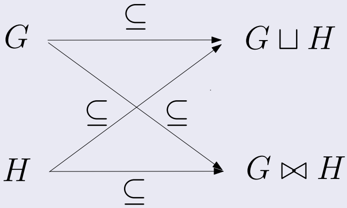
Fact 1

The set Δ_m of equivalence classes of the m-similarity relation is finite. Further, there is a computable function $\Lambda : \mathbb{N} \rightarrow \mathbb{N}$ such that $|\Delta_m| \leq \Lambda(m)$.

Some model-theoretic facts

Fact 2

Each of $\sqcup$ and $\bowtie$ satisfies monotonicity properties.



Some model-theoretic facts

Fact 2

Each of $\sqcup$ and $\bowtie$ satisfies monotonicity properties.

$$
\begin{array}{ccccc}
G_1 & H_1 & \longrightarrow & G_1 \sqcup H_1 & G_1 \bowtie H_1 \\
\downarrow \subseteq & \downarrow \subseteq & & \downarrow \subseteq & \downarrow \subseteq \\
G_2 & H_2 & \longrightarrow & G_2 \sqcup H_2 & G_2 \bowtie H_2
\end{array}
$$

Some model-theoretic facts

Fact 3

Each of $\sqcup$ and $\bowtie$ satisfies a Feferman-Vaught kind composition property.

$$
\begin{array}{ccccc}
G_1 & H_1 & \longrightarrow & G_1 \sqcup H_1 & G_1 \bowtie H_1 \\
\updownarrow \equiv_m & \updownarrow \equiv_m & & \updownarrow \equiv_m & \updownarrow \equiv_m \\
G_2 & H_2 & \longrightarrow & G_2 \sqcup H_2 & G_2 \bowtie H_2
\end{array}
$$

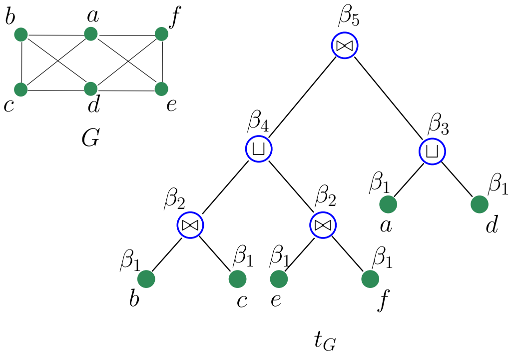
Some model-theoretic facts

Fact 3

Feferman-Vaught kind composition property of $\sqcup$ and $\bowtie$:
There exist composition functions $f_m, g_m : (\Delta_m \times \Delta_m) \rightarrow \Delta_m$ such that if $\delta_m(G)$ is the m-similarity class of G, then

$$\begin{aligned} \delta_m(G_1 \sqcup G_2) &= f_m\big(\delta_m(G_1), \delta_m(G_2)\big) \\ \delta_m(G_1 \bowtie G_2) &= g_m\big(\delta_m(G_1), \delta_m(G_2)\big) \end{aligned}$$

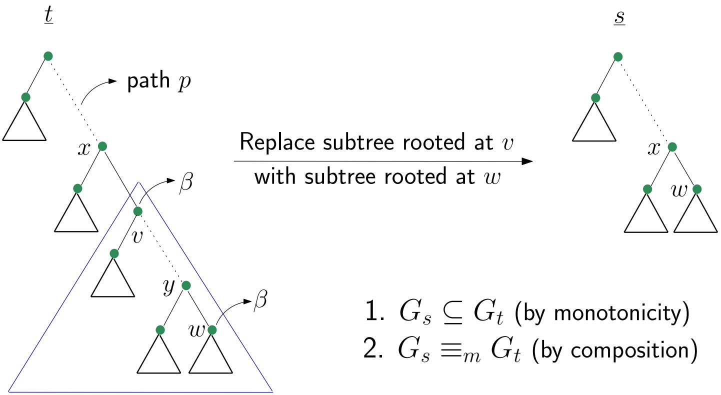
Step I: Label bottom up in the cotree, each node z with the m-similarity class of the graph represented by the tree rooted at z.



$$\beta_1 = \delta_m(\bullet)$$
$$\beta_2 = g_m(\beta_1, \beta_1) = \delta_m(\bullet\!-\!\bullet)$$
$$\beta_3 = f_m(\beta_1, \beta_1) = \delta_m(\bullet \quad \bullet)$$
$$\beta_4 = f_m(\beta_2, \beta_2) = \delta_m\left(\begin{smallmatrix}\bullet & \bullet\\ | & |\\ \bullet & \bullet\end{smallmatrix}\right)$$
$$\beta_5 = g_m(\beta_3, \beta_4) = \delta_m(G)$$

Proof of EBSP for cographs

Step II: Perform graftings in the cotree whenever a root-to-leaf path has repeated labels.



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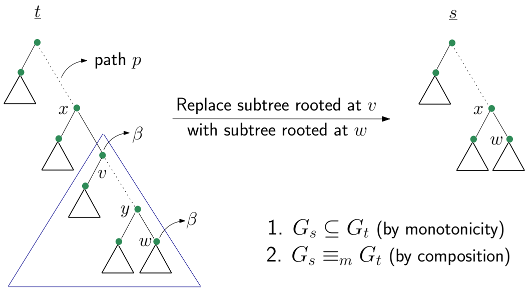

1. $G_s \subseteq G_t$ (by monotonicity)
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Iterate to get a "rainbow" subtree in which no root-to-leaf path has repeated labels. This subtree represents the desired substructure. □

Algorithmic meta-theorems for EBSP classes

- The described technique works for any class of structures that admits "good" tree representations – those which use operations that satisfy monotonicity and composition.
- The composition functions can be computed for any m.
- For any structure and any m, the rainbow subtree can be obtained in time linear in the size of the tree representation of the structure. This subtree represents a small uniform kernel for all FO $[m]$ properties of the original structure.

Theorem

Let $\mathcal{S}$ be a class of structures admitting good tree representations. Then there exists a linear time f.p.t. algorithm for FO model checking over $\mathcal{S}$, provided input structures are given in the form of their tree representations.

Connection with fractals

Fractals

- Mathematical objects that exhibit self-similarity at all scales.
- Appear widely in Nature.

Fern leaf

Fractals

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Conch shell

Fractals

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Romanesco cauliflower

A strengthening of EBSP

Definition

Given a class $\mathcal{S}$ of finite structures, we say $\mathsf{EBSP}^{\#}(\mathcal{S})$ holds if

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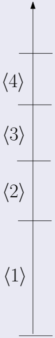

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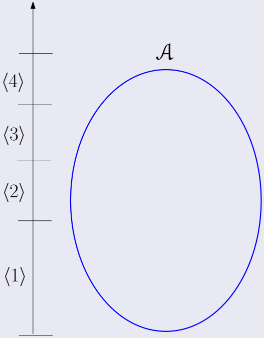

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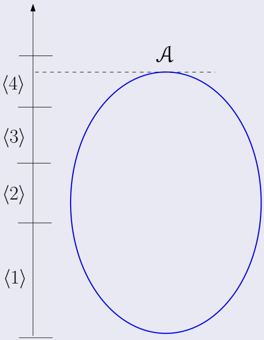

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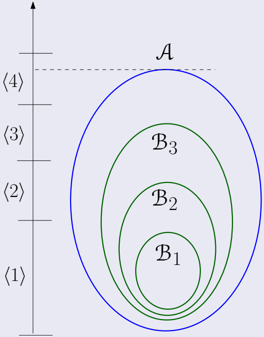

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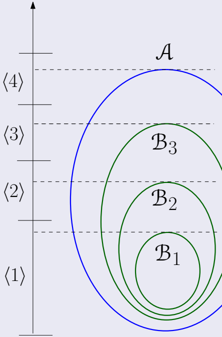

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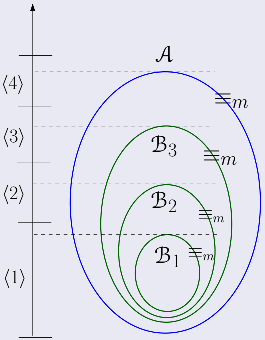

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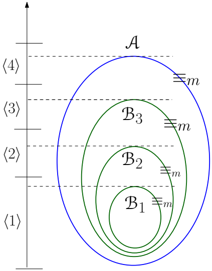
EBSP$^{\#}$ – a fractal-like property



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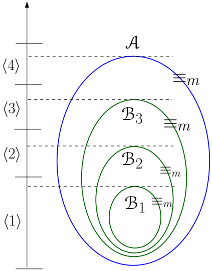

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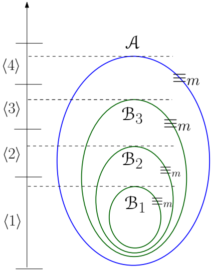

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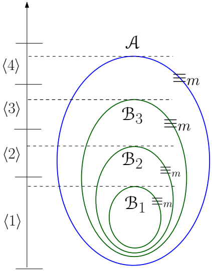

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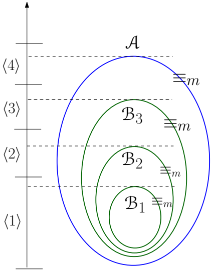

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- All the classes seen so far can be shown to satisfy EBSP$^{\#}$.
- Whereby all these classes can be regarded as logical fractals!

Conclusion

Summary of the talk

- EBSP provides a unifying framework to study a diverse spectrum of interesting classes of finite structures.
- EBSP remains preserved under a variety of natural operations on structures.
- Our techniques used to prove EBSP provide a unified approach for obtaining algorithmic meta-theorems for several interesting classes.
- EBSP has a natural strengthening to a logical fractal property that is enjoyed by all EBSP classes we have investigated.
- The downward Löwenheim-Skolem theorem is strongly prevalent in computer science!

Open questions

- Can we prove a finitary compactness theorem for EBSP classes? And go further towards a Lindström's theorem too?
- What classes of structures satisfy variants of EBSP in which the "substructure" is replaced with other relations (subgraph, homomorphic embedding, minor, etc.)?
- Under what conditions is the index of the m-similarity relation over the class, an elementary function of m?
- Is there a structural characterization of EBSP/logical fractals?
- What classes of structures admit the EBSP/logical fractal property "with high probability"?
- Can we create logical versions of fractal concepts such as fractal dimension, renormalization, etc.?

Tack så mycket!

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