A Generalization of the Łoś-Tarski Preservation Theorem over Classes of Finite Structures

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Joint work with
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Introduction

- Preservation theorems have been one of the earliest areas of study in classical model theory.
- A preservation theorem characterizes (definable) classes of structures closed under a given model theoretic operation.
- Preservation under substructures/extensions (Łoś-Tarski theorem), unions of chains, homomorphisms, etc.
- Most preservation theorems fail in the finite.
- Recent research (by Atserias, Dawar, Grohe, Kolaitis) has focussed on "recovering" preservation results over special classes of finite structures, like acyclic structures, those with bounded degree, bounded tree-width etc.

Talk outline: background and central question

Background (Prior work over arbitrary structures):

- The classical Łoś-Tarski preservation theorem: $PS = \forall^*$
- Preservation under substructures modulo k-cruxes ($PSC(k)$)
- Our generalization of the Łoś-Tarski theorem: $PSC(k) = \exists^k \forall^*$

Central question for this work

What classes $\mathcal{S}$ of finite structures satisfy $PSC(k) = \exists^k \forall^*$?

Talk outline: present work

- A logic-based combinatorial property $\mathcal{P}_{logic}(\mathcal{S}, k)$
- $\mathcal{P}_{logic}(\mathcal{S}, k)$ ensures $PSC(k) = \exists^k \forall^*$ over $\mathcal{S}$
- Interesting classes satisfying $\mathcal{P}_{logic}(\mathcal{S}, k)$
  - Graphs of bounded tree-depth
  - Poset trees
  - Co-graphs
- $\mathcal{P}_{logic}(\mathcal{S}, k)$ and well-quasi-orders
- Conclusion

Some assumptions and notation for the talk

Assumptions:

- First Order (FO) logic.
- Relational vocabularies (i.e. only predicates).
- Familiarity with Ehrenfeucht-Fraïssé games.

Notations:

- $\forall^* = \forall x_1 \ldots \forall x_n$(quantifier-free formula in $x_1, \ldots x_n$)
- $\exists^k\forall^* = \exists x_1 \ldots \exists x_k \forall y_1 \ldots \forall y_n$(quantifier-free formula in $x_1, \ldots, x_k, y_1, \ldots, y_n$)
- $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ means $\mathfrak{A}_1$ is a substructure of $\mathfrak{A}_2$. For graphs, $\subseteq$ means *induced subgraph*.
- $\mathrm{U}_{\mathfrak{A}}$ = universe of $\mathfrak{A}$.

Background

The Łoś-Tarski theorem

Definition 1 (Preservation under substructures)

Let $\mathcal{U}$ be a given class of structures. A sentence ϕ is said to be preserved under substructures over $\mathcal{U}$, abbreviated ϕ is PS over $\mathcal{U}$, if for each structure $\mathfrak{A}$ of $\mathcal{U}$, we have
$\big((\mathfrak{A} \models \phi) \wedge (\mathfrak{B} \subseteq \mathfrak{A}) \wedge (\mathfrak{B} \in \mathcal{U})\big) \rightarrow (\mathfrak{B} \models \phi)$.

- E.g.: Let $\mathcal{U}$ = class of all undirected graphs. Let ϕ describe all cliques, i.e. $\phi = \forall x \forall y E(x, y)$. Then ϕ is PS over $\mathcal{U}$.
- In general, every $\forall^*$ sentence is PS.

Theorem 1 (Łoś-Tarski, 1949-50)

Over the class of all structures, $PS = \forall^*$

Preservation under substructures modulo k-cruxes

Definition 2

Let $\mathcal{U}$ be a given class of structures. A sentence ϕ is said to be preserved under substructures modulo k-cruxes over $\mathcal{U}$, abbreviated ϕ is $PSC(k)$ over $\mathcal{U}$, if for each structure $\mathfrak{A}$ in $\mathcal{U}$, if $\mathfrak{A} \models \phi$, then there is a subset C of $\mathsf{U}_{\mathfrak{A}}$, of size $\leq k$, s.t.
$\big((\mathfrak{B} \subseteq \mathfrak{A}) \wedge (C \subseteq \mathsf{U}_{\mathfrak{B}}) \wedge (\mathfrak{B} \in \mathcal{U})\big) \rightarrow \mathfrak{B} \models \phi$.

- The set C is called a k-crux of $\mathfrak{A}$ w.r.t. ϕ. If ϕ is clear from context, we will call C as a k-crux of $\mathfrak{A}$.
- Easy to see that $PS = PSC(0)$.

Example

- Eg. Consider $\phi = \exists x \forall y E(x, y)$.

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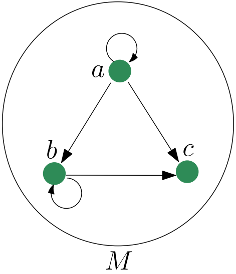

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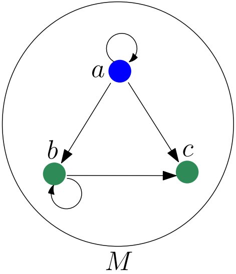

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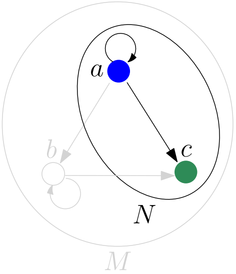

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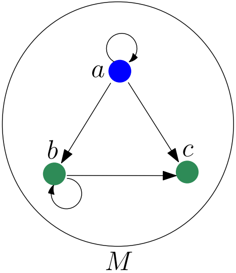

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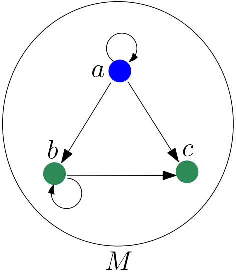

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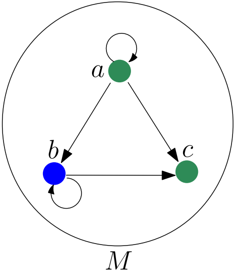

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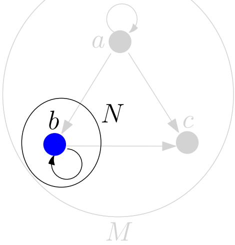

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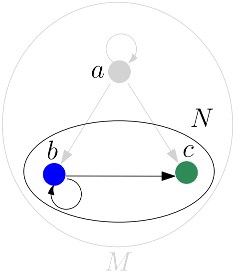

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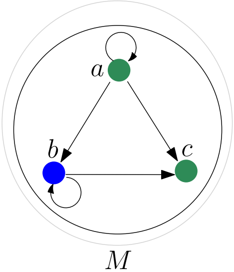

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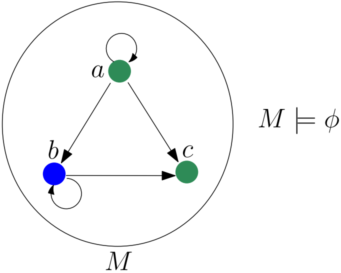

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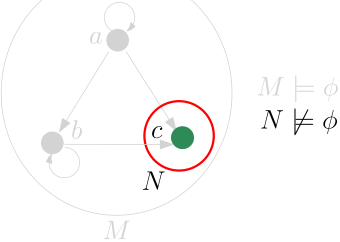

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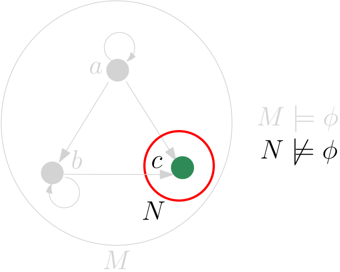

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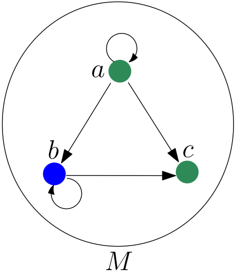

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A generalization of the Łoś-Tarski theorem

- Any $\exists^k \forall^*$ sentence ϕ is $PSC(k)$ – the witnesses to the $\exists$ quantifiers of ϕ form a k-crux.

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Theorem 2

Over the class of all structures, $PSC(k) = \exists^k \forall^*$.

- Note: Bootstrapping on Łoś-Tarski theorem does not work.
- Main idea of proof: prove a dual formulation of the result.
- The proof crucially uses α-saturated structures for $\alpha \geq \omega$ (unlike Łoś-Tarski theorem which requires plain compactness).

What happens in the finite?

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Recall that the Łoś-Tarski theorem

- fails over the class of all finite structures.
- holds over special classes of finite structures like those that are acyclic, of bounded degree or bounded tree-width (under some closure assumptions).

Failure of $PSC(k) = \exists^k\forall^*$ in the finite

Proposition 1

Over the class of all finite structures, $PSC(k) \supsetneq \exists^k\forall^*$ for all k.

Proposition 2

$\mathcal{U} =$ *class of disjoint unions of undirected paths*

$\phi =$ "$(\exists_{\geq 3} x\ deg(x) \leq 1) \bigvee (\exists_{\geq 2} x\ deg(x) = 0)$"

Then ϕ is $PSC(2)$ but $\phi \not\leftrightarrow \exists^2\forall^*(\ldots)$, over $\mathcal{U}$.

- $\mathcal{U}$ is acyclic, of degree ≤ 2, and closed under substructures and disjoint unions. A similar failure can be shown over the class of bounded tree-width graphs.
- Summary: The classes identified by Atserias, Dawar and Grohe satisfy $PS = \forall^*$ but not $PSC(k) = \exists^k\forall^*$ for $k \geq 2$.

Motivation of this work

Central question

Can we identify abstract structural properties of classes of finite structures, that are satisfied by interesting classes, and that admit $PSC(k) = \exists^k \forall^*$?

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Can we identify abstract structural properties of classes of finite structures, that are satisfied by interesting classes, and that admit $PSC(k) = \exists^k \forall^*$? And further, in effective form?

A logic based combinatorial property

Definition 3

Given a class $\mathcal{S}$ *and* $k \in \mathbb{N}$*, we say that* $\mathcal{P}_{logic}(\mathcal{S}, k)$ *holds if there exists a function* $\theta_k : \mathbb{N} \to \mathbb{N}$ *such that*

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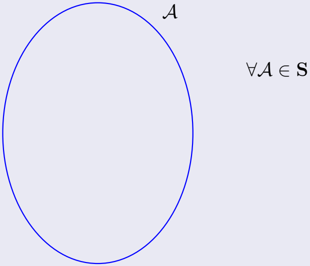

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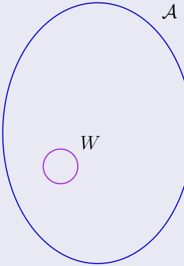

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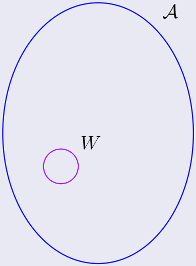

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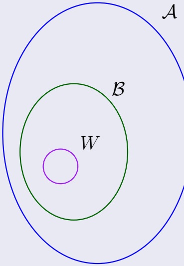

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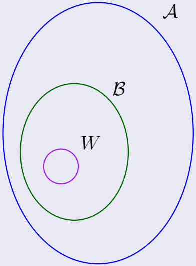

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(i) $\mathcal{B} \in \mathcal{S}$ (ii) $\mathcal{B} \equiv_m \mathcal{A}$
(iii) $|\mathcal{B}| \leq \theta_k(m)$

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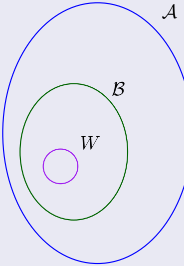

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We call θ_k a witness function of $\mathcal{P}_{logic}(\mathcal{S}, k)$.

$\mathcal{P}_{logic}(\mathcal{S}, k)$ and $PSC(k) = \exists^k \forall^*$

- E.g.: Let $\mathcal{S}$ = class of all finite linear orders.
- Any two linear orders of size 2^m are m-equivalent.
- Then $\mathcal{P}_{logic}(\mathcal{S}, k)$ holds for all k, where $\theta_k(m) = \max(2^m, k)$.

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Theorem 3

Let $\mathcal{S}$ be a class of finite structures and $k \in \mathbb{N}$ be such that $\mathcal{P}_{logic}(\mathcal{S}, k)$ holds. Then $PSC(k) = \exists^k\forall^*$ over $\mathcal{S}$.
Furthermore, if the witness function is computable, then so is the translation from $PSC(k)$ to $\exists^k\forall^*$.

Graphs of bounded tree-depth satisfy $\mathcal{P}_{logic}(\cdot, k)$

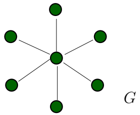
Graphs of bounded tree-depth

- Nešetřil and de Mendez introduced the notion of *tree-depth* of an undirected graph.
- Intuitively, the tree-depth of a graph G, denoted $td(G)$, is a measure of how far G is from being a star.
- Formally, if $G = (V, E)$ and $\mathsf{Comp}(G)$ = connected components of G, then

$$td(G) = \begin{cases} 1 & \text{if } G \text{ is a single node} \\ \max_{G' \in \mathsf{Comp}(G)} td(G') & \text{if } G \text{ is disconnected} \\ 1 + \min_{v \in V} td(G \setminus v) & \text{otherwise} \end{cases}$$

Examples

1) Star



$td(G) = 2$

Examples

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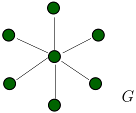

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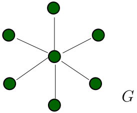

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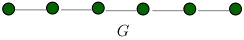
2) Path



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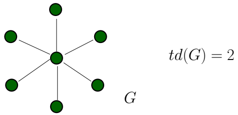
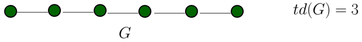


2) Path : Tree-depth of a d length path $\approx log_2(d)$.



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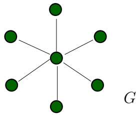

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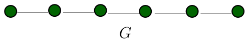

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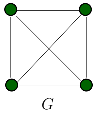
3) Cliques



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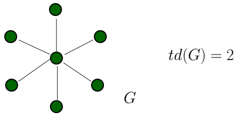

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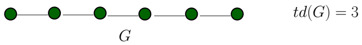
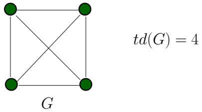


3) Cliques : Cliques have unbounded tree-depth.



Bounded tree-depth graphs satisfy $\mathcal{P}_{logic}(\cdot, k)$

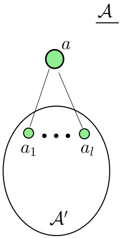
Theorem 4

Given $n \in \mathbb{N}$, let $\mathcal{S}$ be any class of graphs having tree-depth $\leq n$, that is closed under induced subgraphs. Then $\forall k \mathcal{P}_{logic}(\mathcal{S}, k)$ holds. Further, there exists a computable witness function.

- Bounded tree-depth classes also have bounded tree-width.
- Atserias, Dawar and Grohe showed that for each $n \in \mathbb{N}$, the class of all graphs of tree-width $\leq n$ satisfies the Łoś-Tarski theorem, and that in general, subclasses of this class do not satisfy the Łoś-Tarski theorem.
- However Theorem 4 identifies for each $n \in \mathbb{N}$, an important subclass of tree-width $\leq n$ graphs, satisfying not only the Łoś-Tarski theorem, but also an effective generalization of it.

Proof Idea for Theorem 4

$\tau = \{E, U, p\}$

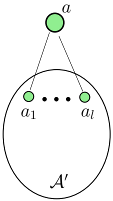


$U^{\mathcal{A}} = \{a, \ldots\}$
$p^{\mathcal{A}} =$ False

$td(\mathcal{A}) = n$
$td(\mathcal{A}') < n$

Proof Idea for Theorem 4

$\mathcal{A}$ $\quad \tau = \{E, U, p\}$

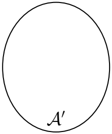


$U^{\mathcal{A}} = \{a, \ldots\}$
$p^{\mathcal{A}} = \text{False}$

$td(\mathcal{A}) = n$
$td(\mathcal{A}') < n$

twin ⟷

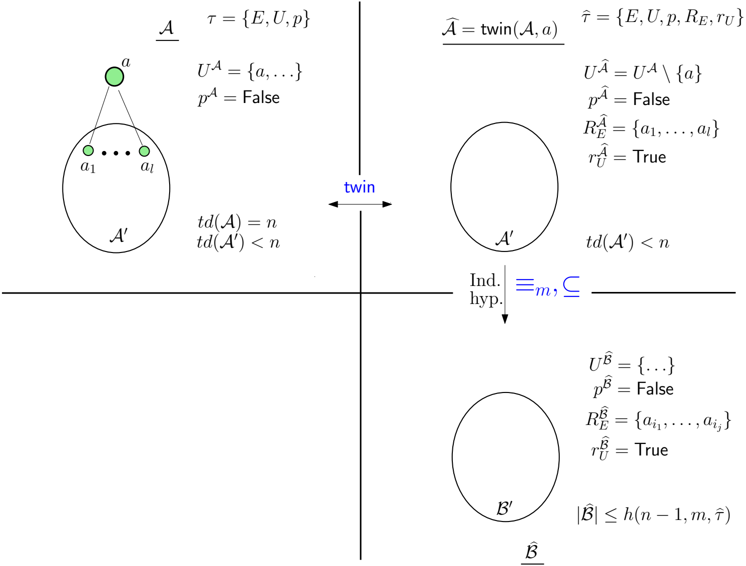
$\widehat{\mathcal{A}} = \text{twin}(\mathcal{A}, a)$ $\quad \hat{\tau} = \{E, U, p, R_E, r_U\}$



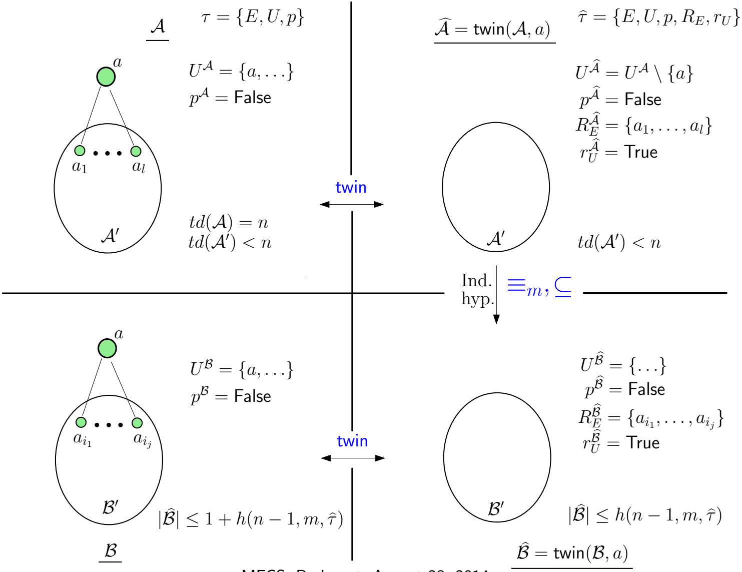
$U^{\widehat{\mathcal{A}}} = U^{\mathcal{A}} \setminus \{a\}$
$p^{\widehat{\mathcal{A}}} = \text{False}$
$R_E^{\widehat{\mathcal{A}}} = \{a_1, \ldots, a_l\}$
$r_U^{\widehat{\mathcal{A}}} = \text{True}$

$td(\mathcal{A}') < n$

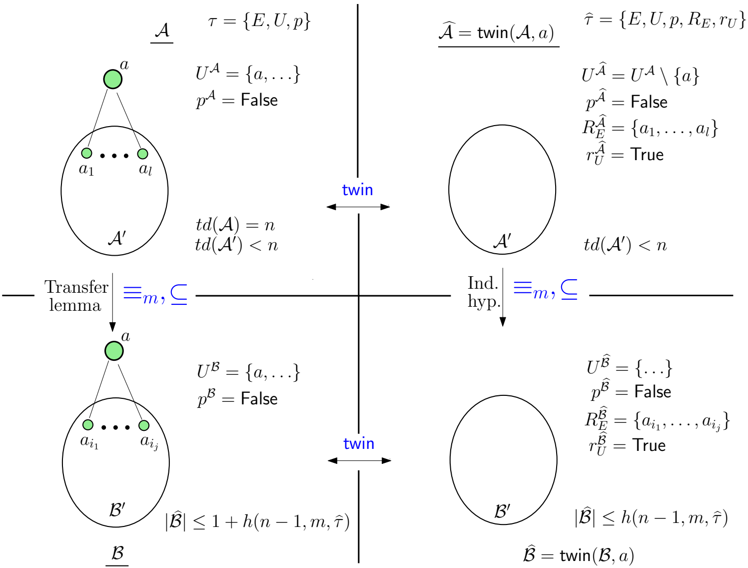
Proof Idea for Theorem 4



Proof Idea for Theorem 4



Proof Idea for Theorem 4



Classes of unbounded tree-depth satisfying $\mathcal{P}_{logic}(\cdot, k)$

Structures of unbounded tree-depth satisfying $\mathcal{P}_{logic}(\mathcal{S}, k)$ - Σ-trees

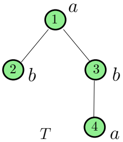
- A Σ-tree is finite poset tree labeled with an alphabet Σ.

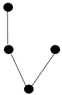
$T = (P, \lambda)$

$P = \begin{matrix} \{(1,1),(2,2),(3,3),(4,4)\} \cup \\ \{(1,2),(1,3),(3,4),(1,4)\}\} \end{matrix}$

$\lambda = \{(1,a),(2,b),(3,b),(4,a)\}\}$

Node 1 is the root of T





Hasse diagram of T

- Let Trees(Σ) = class of all Σ-trees. Note: $td(\text{Trees}(\Sigma)) = \omega$.

Theorem 5

$\mathcal{P}_{logic}(\text{Trees}(\Sigma), k)$ *holds for each finite alphabet* Σ *and* $k \in \mathbb{N}$. *Further, there exists a computable witness function.*

Structures of unbounded tree-depth satisfying $\mathcal{P}_{logic}(\mathcal{S}, k)$ - Co-graphs

- Co-graphs are defined inductively:
  - Base case: isolated vertices
  - Induction: If G_1 and G_2 are co-graphs, then so is the (i) disjoint union $G_1 \sqcup G_2$ and (ii) complement $\overline{G}_1$.
- Examples of co-graphs include cliques, n-partite graphs, Turán graphs, etc.
- If CG = class of all co-graphs, then $td(\text{CG}) = \omega$ as $K_n \in \text{CG}$.

Theorem 6

$\mathcal{P}_{logic}(\text{CG}, k)$ *holds for all* k*. Further, there exists a computable witness function.*

$\mathcal{P}_{logic}(\cdot, 0)$ and well-quasi-orders

- A poset $(A, \leq)$ is said to be a *well-quasi-order* (w.q.o.) if for all infinite sequences $a_1, a_2, \ldots$ from A, there exists i, j such that $i < j$ and $a_i \leq a_j$. We say A *is w.q.o. under* $\leq$.
- Eg. words and trees are respectively w.q.o. under (isomorphic) embedding by Higman's lemma and Kruskal's tree theorem.

Theorem 7

If $\mathcal{S}$ is w.q.o. under embedding, then $\mathcal{P}_{logic}(\mathcal{S}, 0)$ holds.

This result gives a "logic based" tool to prove non-w.q.o.-ness of classes of structures under embedding!

Conclusion

Summary:

- A generalization of the Łoś-Tarski theorem: $PSC(k) = \exists^k \forall^*$
- A logic-based combinatorial property $\mathcal{P}_{logic}(\mathcal{S}, k)$, where $\mathcal{S}$ is a class of finite structures and $k \in \mathbb{N}$
- $\mathcal{P}_{logic}(\mathcal{S}, k)$ ensures $PSC(k) = \exists^k \forall^*$ over $\mathcal{S}$
- Interesting classes satisfying $\mathcal{P}_{logic}(\mathcal{S}, k)$
- $\mathcal{P}_{logic}(\mathcal{S}, k)$ and well-quasi-orders

Future work:

Investigate the boundaries of when $PSC(k) = \exists^k \forall^*$ holds over classes of finite structures.

Köszönöm!

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