Hereditariness in the finite and prefix classes of first order logic

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Verification Seminar
University of Oxford, UK

Nov 7, 2019

Introduction

- Hereditariness is a well realized property in computer science. E.g. cliques, bounded degree graphs, 3-colorable graphs, graphs of bounded clique-width, etc.
- The Łoś-Tarski theorem characterizes FO definable hereditary properties in terms of universal sentences.
- Historically significant: among the earliest applications of Gödel's Compactness theorem and opened the area of preservation theorems in model theory.
- Fails in the finite: there is a hereditary FO sentence that is not equivalent to any universal sentence over all finite structures (Gurevich-Shelah, 1984).
- But already in 1959, Tait gave a different counterexample, that turns out to be more powerful than known so far.

Main results

Let $\Sigma_n := \underbrace{\exists \bar{x}_1 \forall \bar{x}_2 \exists \bar{x}_3 \ldots}_{n \text{ blocks}} \alpha(\bar{x}_1, \ldots, \bar{x}_n)$ where α is quantifier-free.

Theorem

Tait's counterexample is an FO sentence that is hereditary over all finite structures, but is not equivalent over this class to any Σ_3 sentence. Further, the negation of the counterexample can be expressed in Datalog$(\neq, \neg)$.

Theorem

For every n, there is a vocabulary τ_n and an FO(τ_n) sentence φ_n that is hereditary over all finite structures, but that is not equivalent over this class to any Σ_n sentence. Further, $\neg\varphi_n$ can be expressed in Datalog$(\neq, \neg)$.

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Theorem

No prefix classes of FO is expressive enough to capture:

- FO-hereditariness in the finite
- FO $\cap$ Datalog$(\neq, \neg)$ queries in the finite

Analysing Tait's sentence

Analysing Tait's sentence: overview

- The sentence
- Showing hereditariness
- Construction of a suitable class of models and non-models
- Showing inexpressibility in Σ_2, and then in Σ_3
- Showing expressibility of negation of the sentence in Datalog($\neq, \neg$)

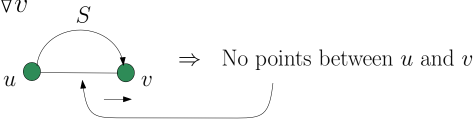
The sentence

Tait's sentence

$$\Psi := \xi_1 \wedge \xi_2 \wedge \xi_3 \quad \in \mathrm{FO}(\sigma) \text{ where } \sigma = \{\leq, R, S\}$$

$\xi_1 :=$ " $\leq$ is a linear order"

$\xi_2 := \forall u \forall v$



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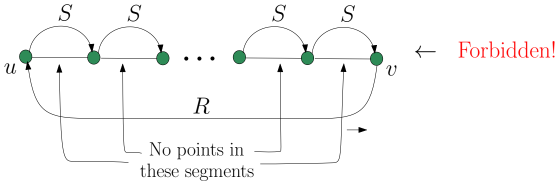
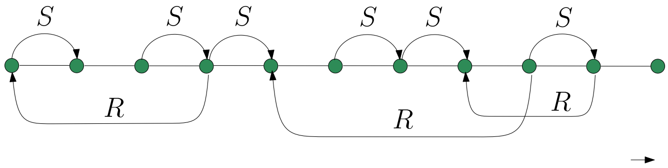


A model for Ψ



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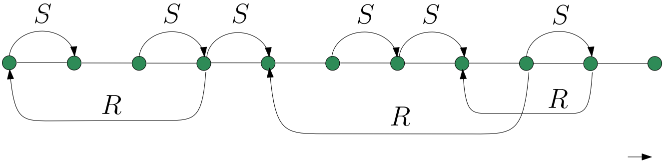

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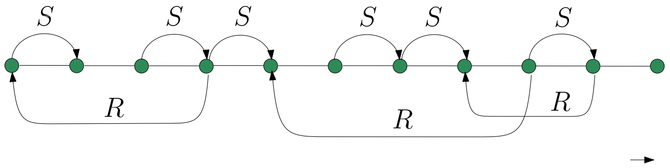

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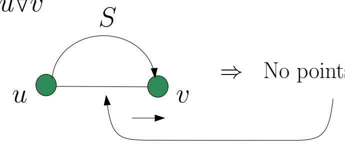

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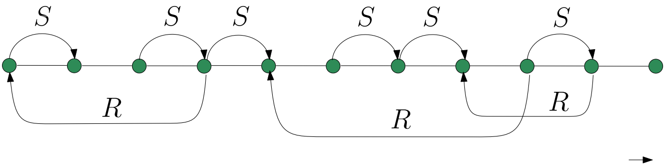

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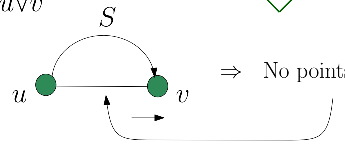

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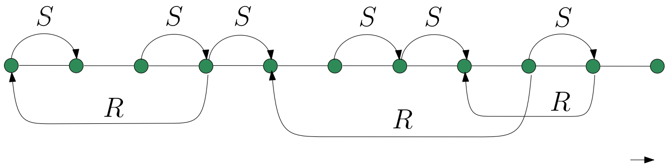

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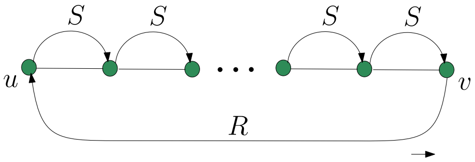

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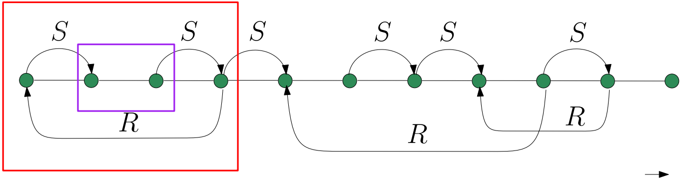

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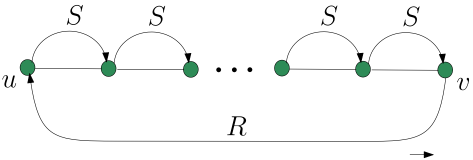

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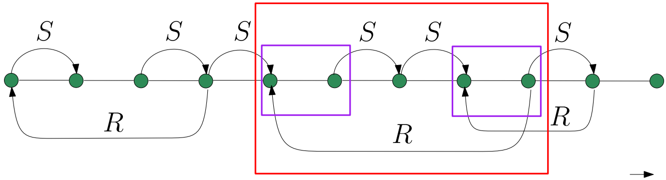

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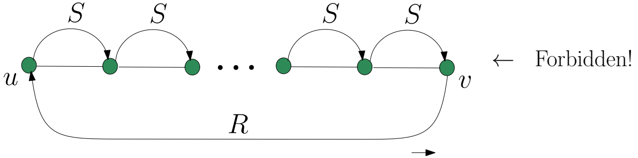

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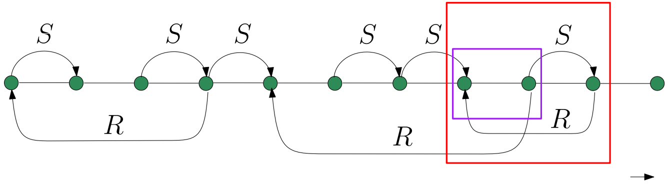

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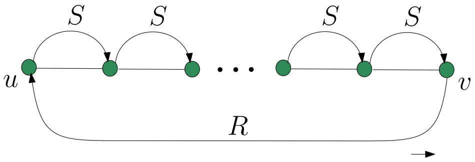

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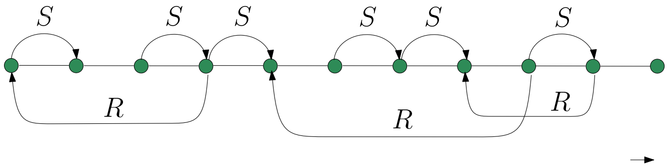

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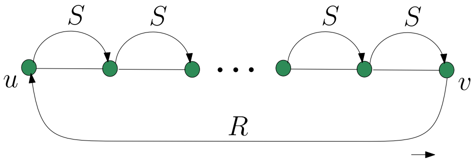

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Hereditariness of Tait's sentence

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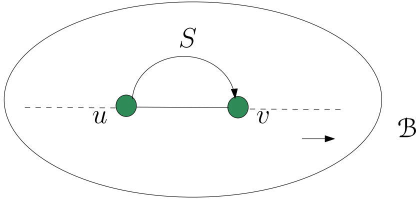

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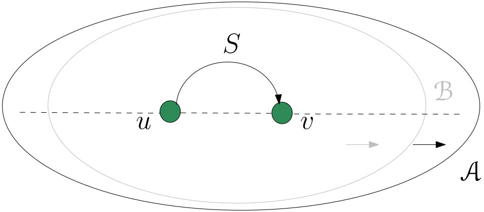

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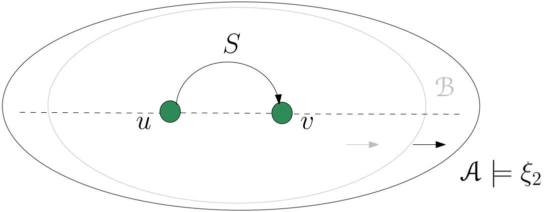

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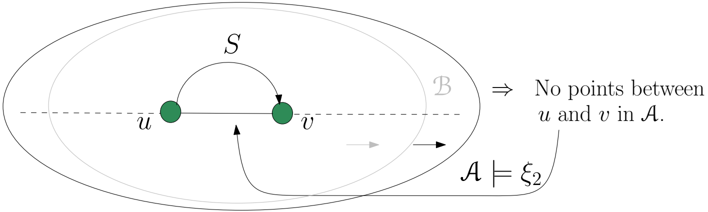

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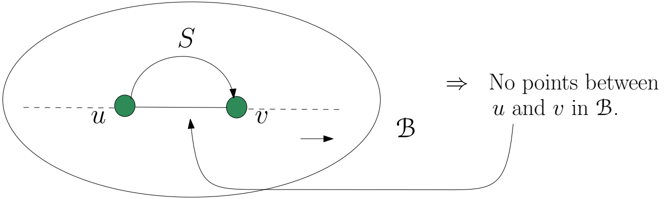

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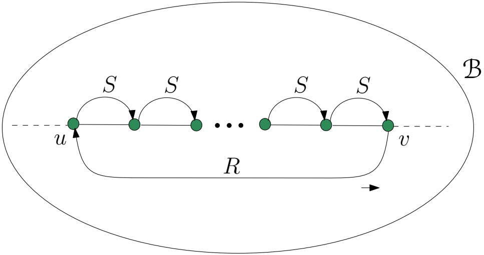

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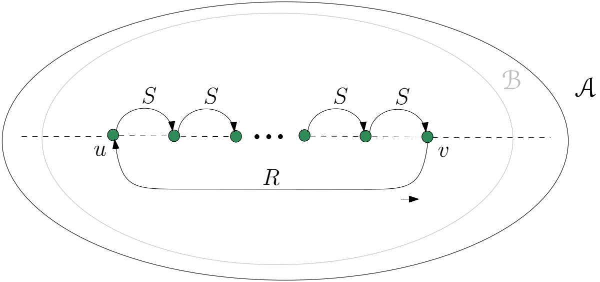

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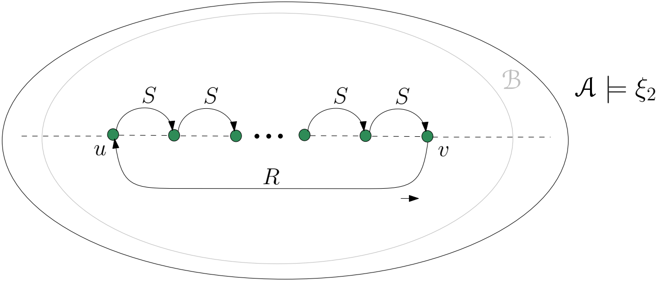

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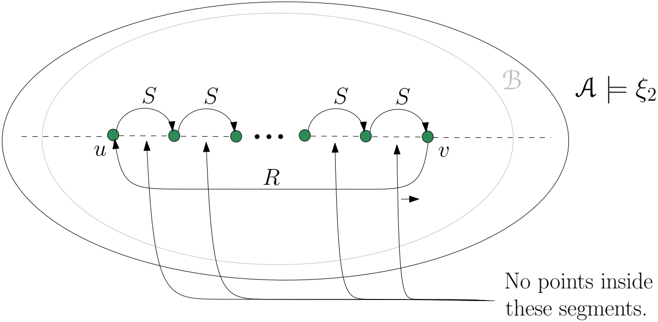

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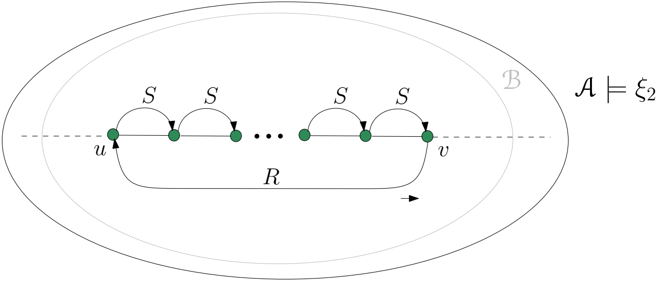

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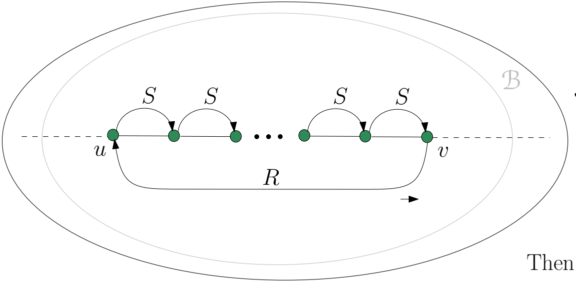

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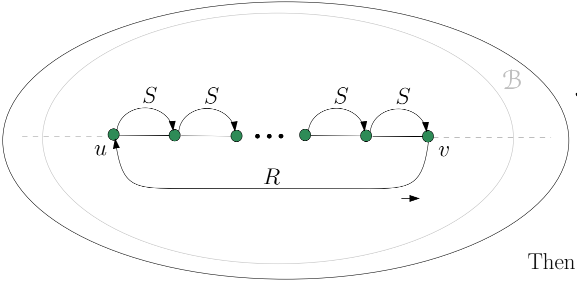
- As explained, $\mathcal{B} \models \xi_2$ and $\mathcal{B} \models \xi_3$; then $\mathfrak{B} \models \Psi$. $\square$



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Stronger failure of Łoś-Tarski theorem in the finite

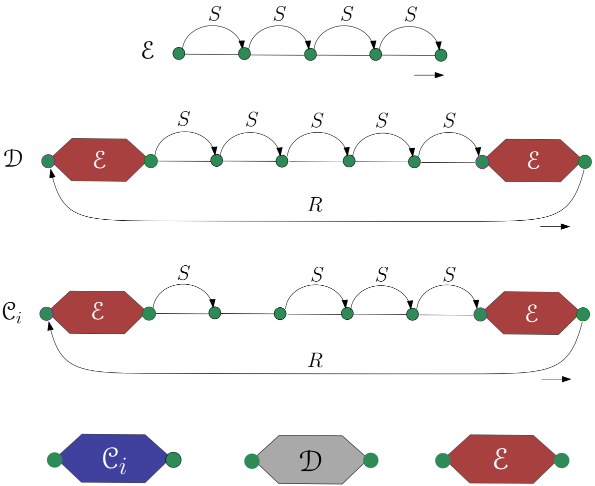
Proposition

The sentence Ψ, which is hereditary over all finite structures, is not equivalent over this class to any Σ_2 sentence.

- Let $\Sigma_{2,k}$ = class of Σ_2 sentences in which each block of quantifiers has size k.
- For each k, we construct a class $\mathbf{A}$ of models and a class $\mathbf{B}$ of non-models of Ψ such that for each $\Sigma_{2,k}$ sentence θ, if there is a model of θ in $\mathbf{A}$, then there is a model of θ in $\mathbf{B}$ as well.
- Denote the above condition as $\mathbf{A} \Rrightarrow_{2,k} \mathbf{B}$.
- We illustrate our constructions for $k = 3$.

Construction of
a class of models and a class of non-models

Construction of classes $\mathbf{A}$ and $\mathbf{B}$



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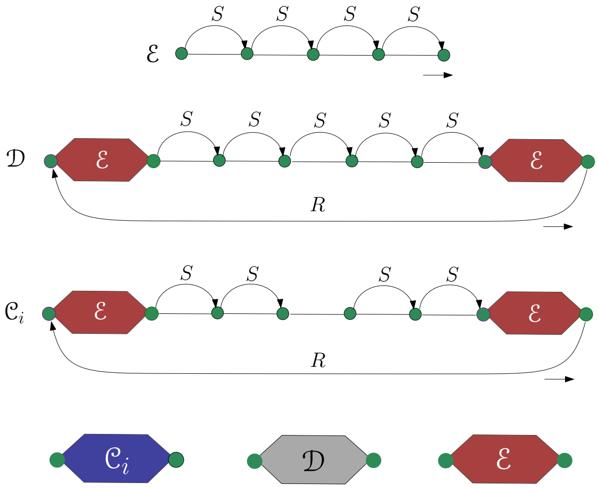

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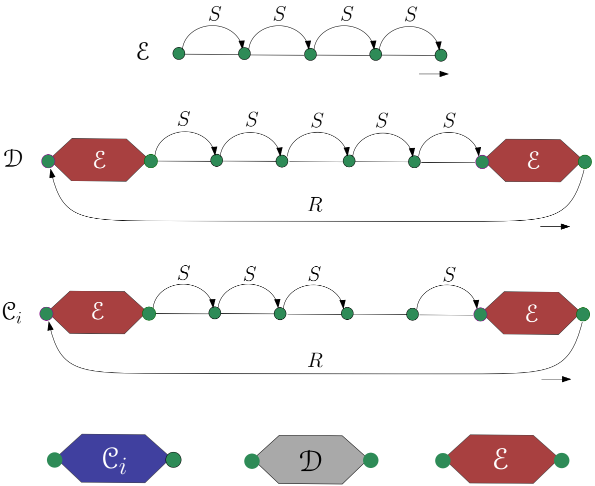

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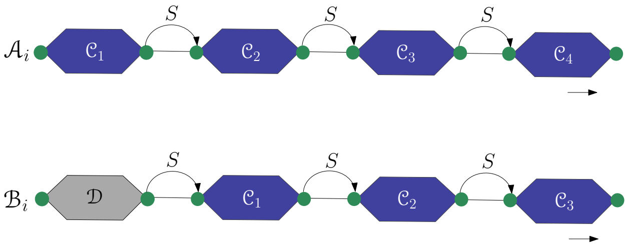

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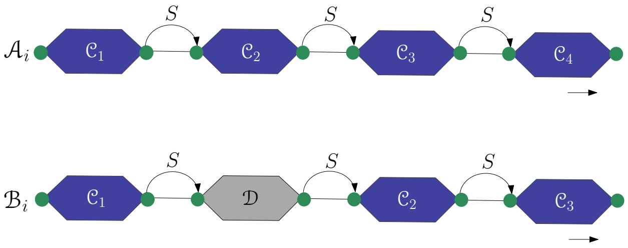

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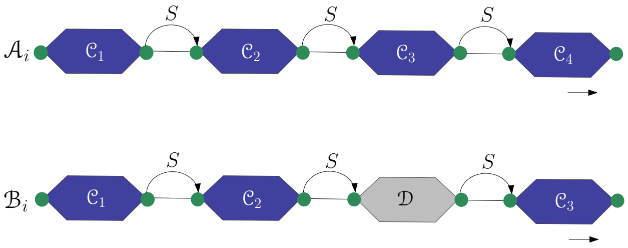

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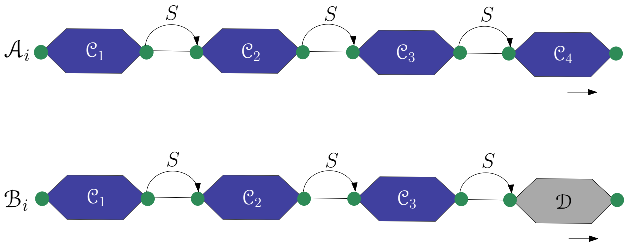
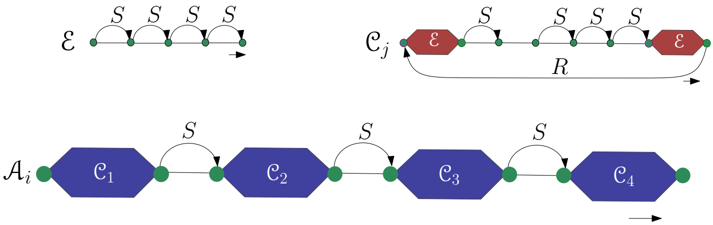


$\mathbf{A}$ = Class of all $\mathcal{A}_i$s; $\mathbf{B}$ = Class of all $\mathcal{B}_i$s

A as a class of models of Ψ



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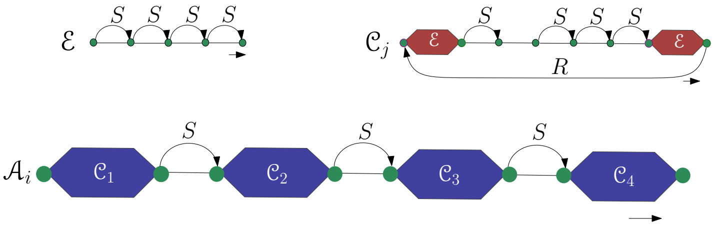

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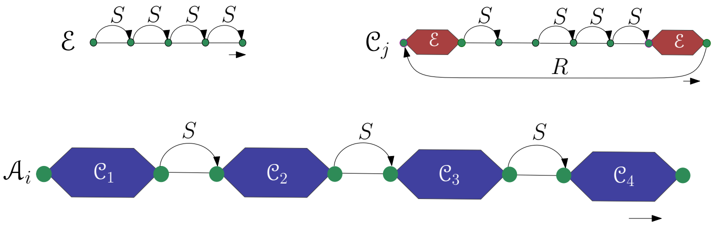

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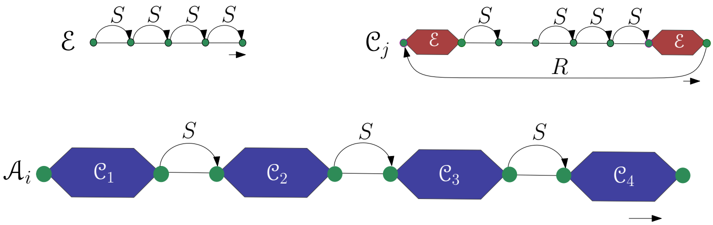

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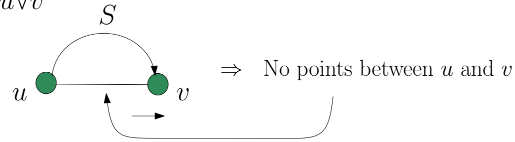

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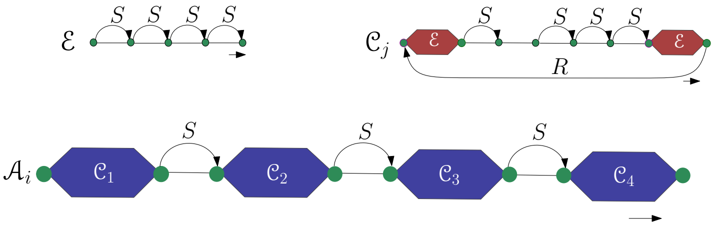

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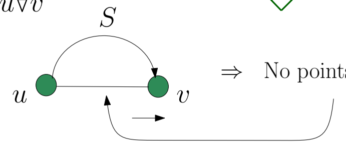
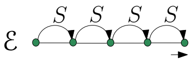
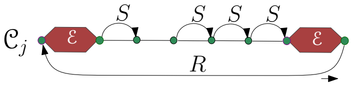
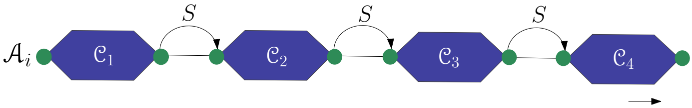


$\Rightarrow$ No points between u and v

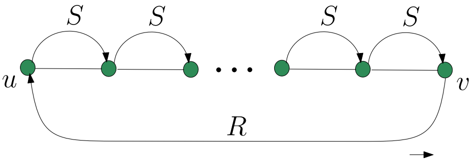
$\mathbb{A}$ as a class of models of Ψ





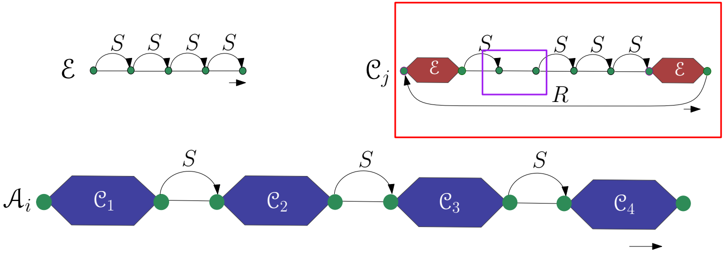


$\xi_3 := \forall u \forall v$



$\leftarrow$ Forbidden!

A as a class of models of Ψ



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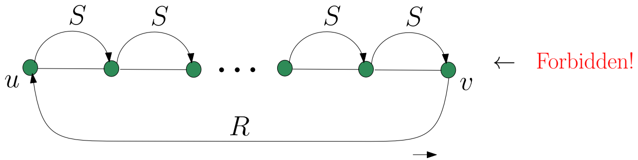
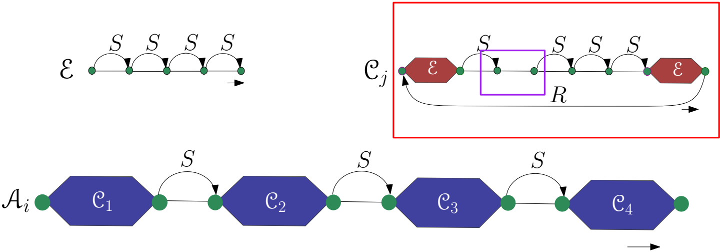
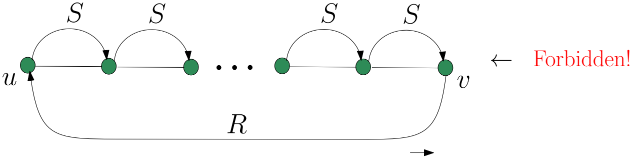


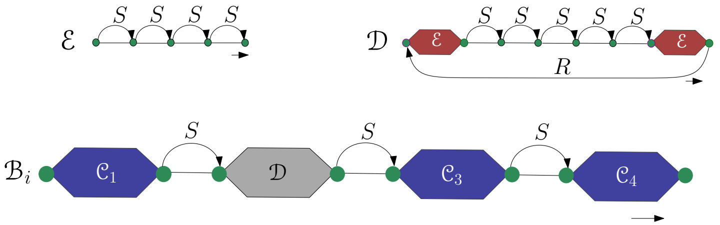
$\mathbb{A}$ as a class of models of Ψ



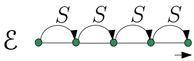
$\xi_3 := \forall u \forall v$

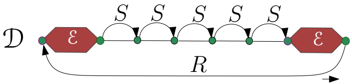


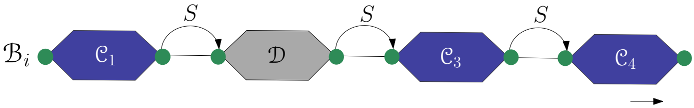
$\mathbb{B}$ as a class of non-models of Ψ



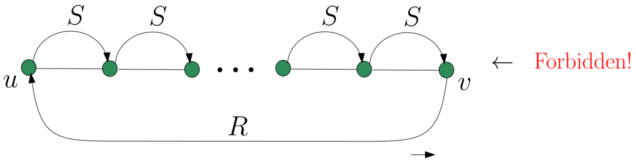
B as a class of non-models of Ψ



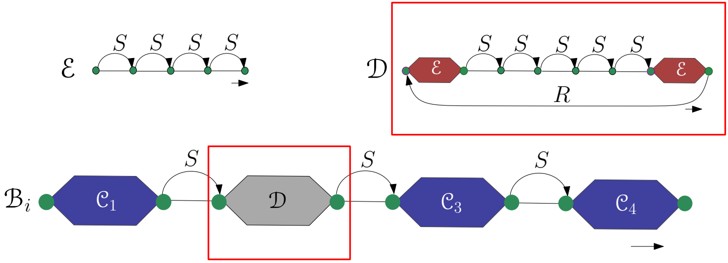




$\xi_3 := \forall u \forall v$



B as a class of non-models of Ψ



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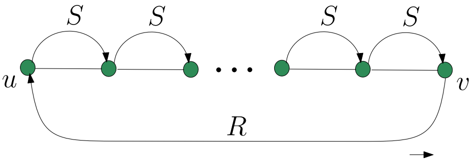
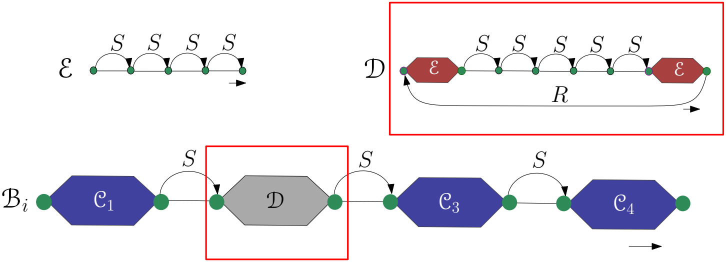


$\leftarrow$ Forbidden!



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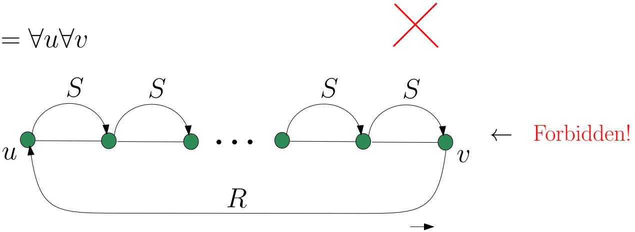


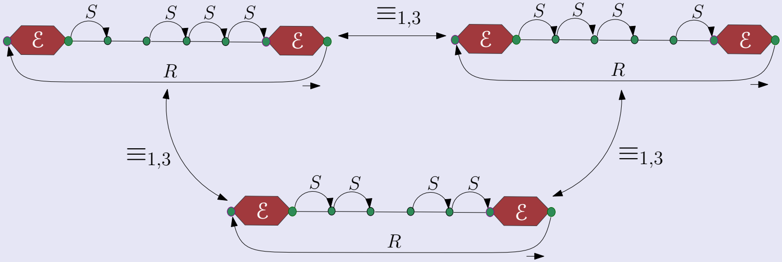
Inexpressibility in Σ_2

Towards showing $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$

- For structures $\mathcal{M}_1$ and $\mathcal{M}_2$, denote by $\mathcal{M}_1 \equiv_{1,3} \mathcal{M}_2$ that $\mathcal{M}_1$ and $\mathcal{M}_2$ agree on all sentences of $\Sigma_{1,3}$.
- The relation $\equiv_{1,3}$ is an equivalence relation (of finite index).
- E.g.: any two linear orders of length ≥ 3 are $\equiv_{1,3}$-equivalent.
- One can build pairs of $\equiv_{1,3}$-equivalent structures from given such pairs using operators on structures that satisfy the Feferman-Vaught (FV) composition property.
- A binary operator $\oplus$ satisfies FV composition w.r.t. $\equiv_{1,3}$ if the $\equiv_{1,3}$-class of $\mathcal{M} \oplus \mathcal{N}$ is completely determined by the $\equiv_{1,3}$-classes of $\mathcal{M}$ and $\mathcal{N}$.
- E.g.: the ordered sum of two (ordered) structures satisfies FV-composition w.r.t. $\equiv_{1,3}$.

Some $\equiv_{1,3}$-equivalences

Lemma

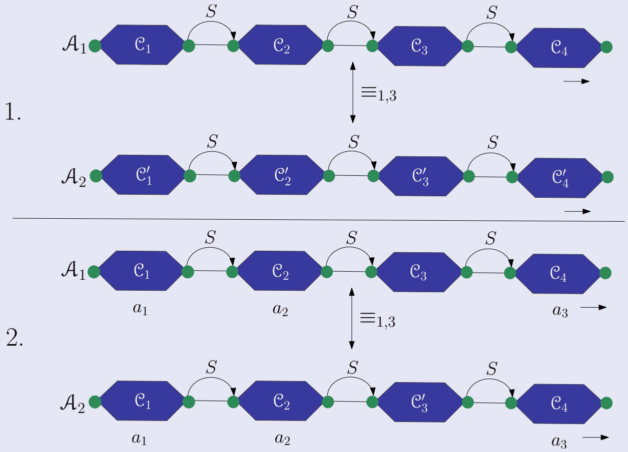


Proof Sketch.

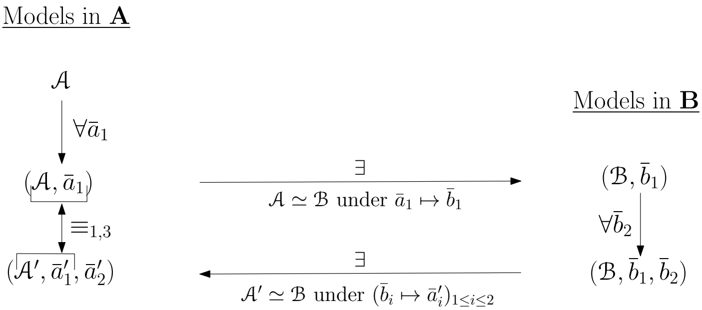
Using FV-composition for the ordered sum of linear orders equipped with a full successor relation and colored endpoints, and the fact that such linear orders of length ≥ 4 are $\equiv_{1,3}$-equivalent. □

Some $\equiv_{1,3}$-equivalences

Corollary

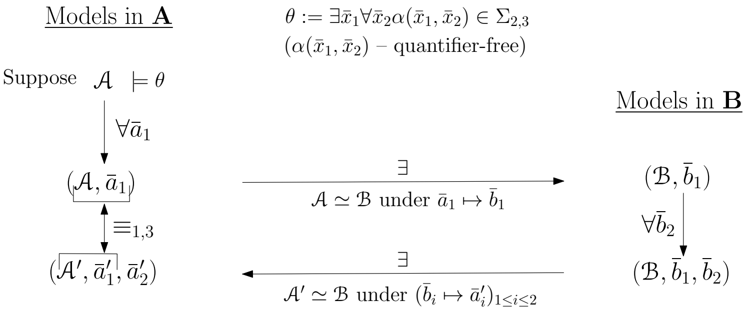


Overview of proof approach for showing $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$



"$\mathcal{A} \simeq \mathcal{B}$ under $\bar{a} \mapsto \bar{b}$" $:=$ $\bar{a} \mapsto \bar{b}$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Overview of proof approach for showing $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$



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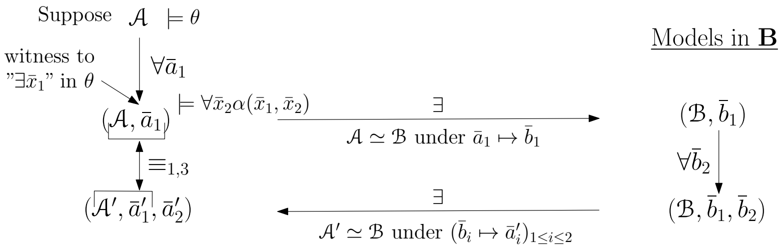
Overview of proof approach for showing $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$

Models in $\mathbf{A}$

$\theta := \exists \bar{x}_1 \forall \bar{x}_2 \alpha(\bar{x}_1, \bar{x}_2) \in \Sigma_{2,3}$
$(\alpha(\bar{x}_1, \bar{x}_2)$ – quantifier-free)



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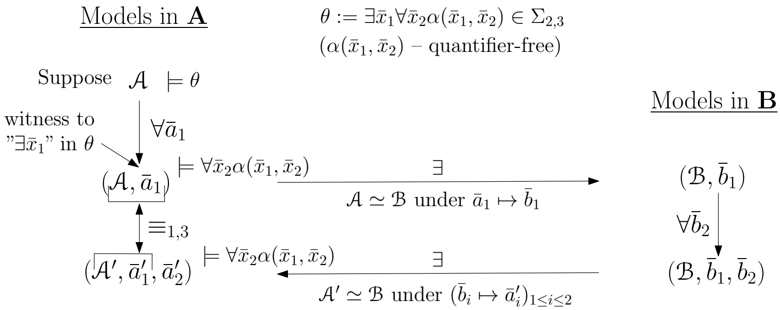
Overview of proof approach for showing $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$



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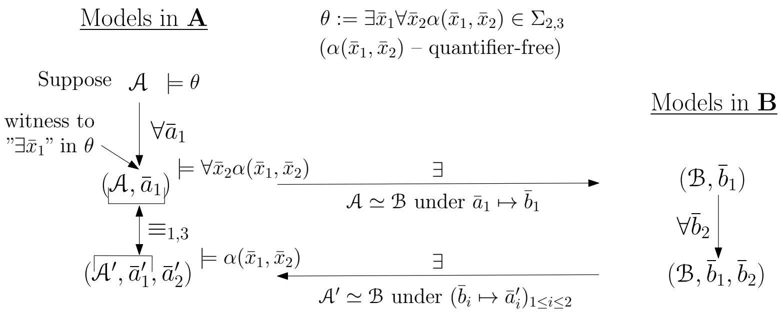
Overview of proof approach for showing $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$



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Overview of proof approach for showing $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$

<u>Models in $\mathbf{A}$</u>

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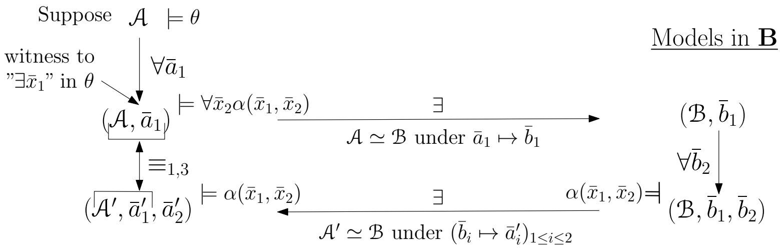
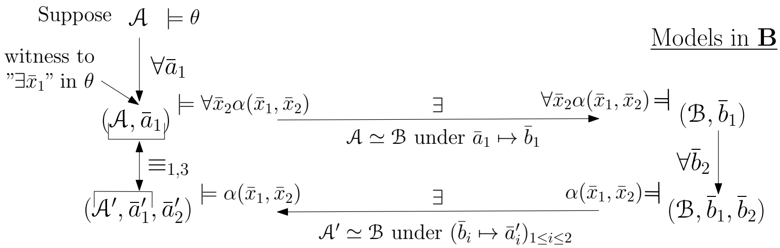


"$\mathcal{A} \simeq \mathcal{B}$ under $\bar{a} \mapsto \bar{b}$" $:= \bar{a} \mapsto \bar{b}$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Overview of proof approach for showing $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$

Models in **A**

$\theta := \exists \bar{x}_1 \forall \bar{x}_2 \alpha(\bar{x}_1, \bar{x}_2) \in \Sigma_{2,3}$
$(\alpha(\bar{x}_1, \bar{x}_2)$ – quantifier-free$)$



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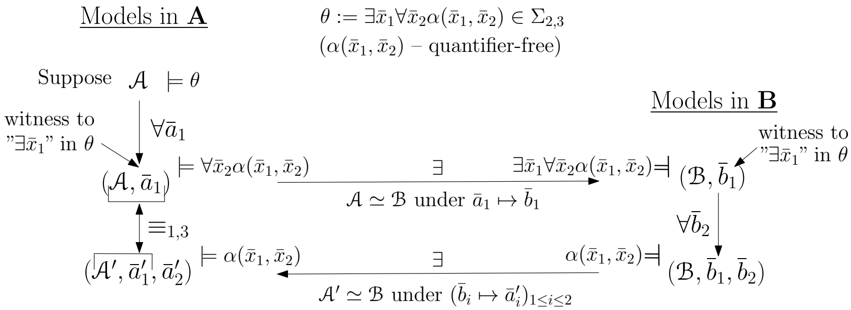
Overview of proof approach for showing $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$



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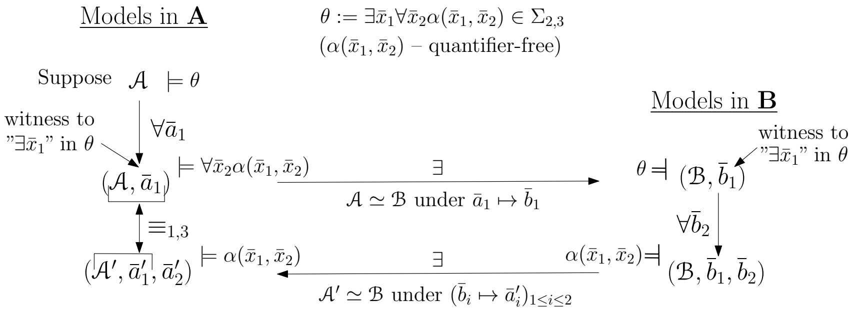
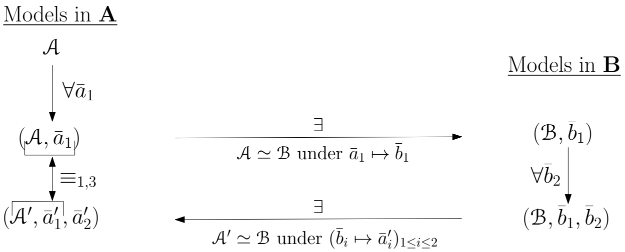


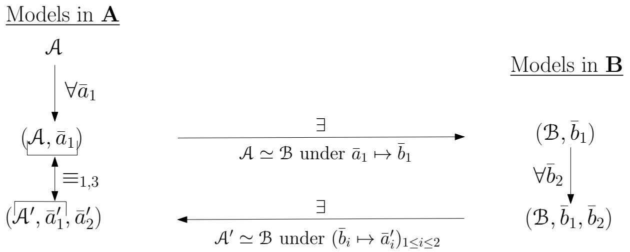
"$\mathcal{A} \simeq \mathcal{B}$ under $\bar{a} \mapsto \bar{b}$" $:= \bar{a} \mapsto \bar{b}$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Proof approach as an Ehrenfeucht-Fräissé game



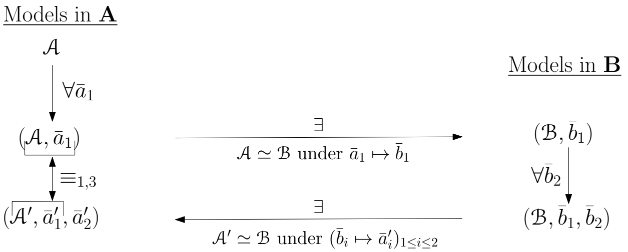
- Players: Spoiler, Duplicator; Game arena: just structure $\mathcal{A}$ initially.
- Round 1: Spoiler picks a 3-tuple $\bar{a}_1$ from $\mathcal{A}$. In response, Duplicator first chooses $\mathcal{B} \in \mathbf{B}$; then picks a 3-tuple $\bar{b}_1$ from $\mathcal{B}$.
- Winning condition: Duplicator wins the round if $\mathcal{A} \simeq \mathcal{B}$ under the map $\bar{a}_1 \mapsto \bar{b}_1$. Else Spoiler wins (this play of) the game.

Proof approach as an Ehrenfeucht-Fräissé game



- Round 2: Spoiler picks a 3-tuple $\bar{b}_2$ from $\mathcal{B}$. In response, Duplicator first chooses $\mathcal{A}' \in \mathbf{A}$; then picks 3-tuples $\bar{a}_1', \bar{a}_2'$ from $\mathcal{A}'$.
- Winning condition: Duplicator wins the round and (this play of) the game if (i) $(\mathcal{A}', \bar{a}_1') \equiv_{1,3} (\mathcal{A}, \bar{a}_1)$; (ii) $\mathcal{A}' \simeq \mathcal{B}$ under $(\bar{a}_i' \mapsto \bar{b}_i)_{1 \leq i \leq 2}$. (Else Spoiler wins.)

Proof approach as an Ehrenfeucht-Fräissé game

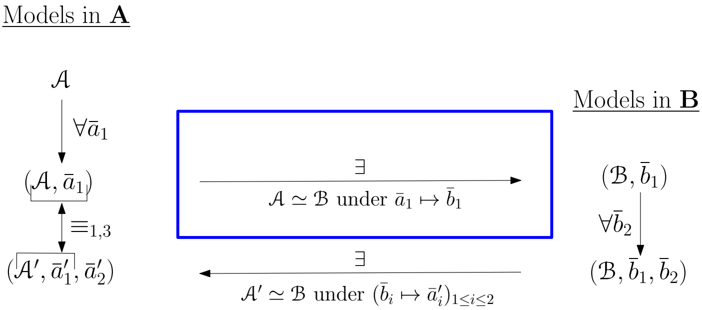


- Duplicator has a winning strategy in the game described if she wins every play of the game.

Proposition

If Duplicator has winning strategy in the described game, then $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$.

Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{A}$ to $\mathbf{B}$



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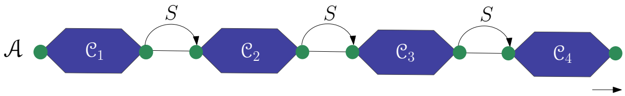
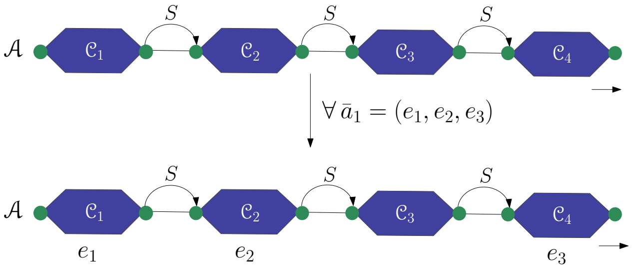
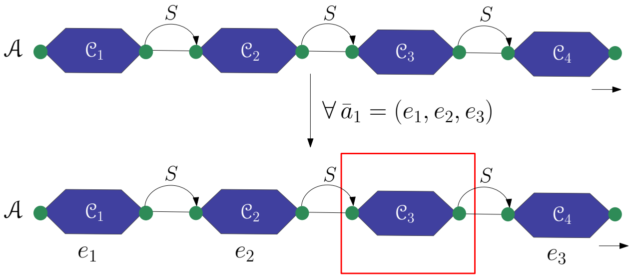
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{A}$ to $\mathbf{B}$



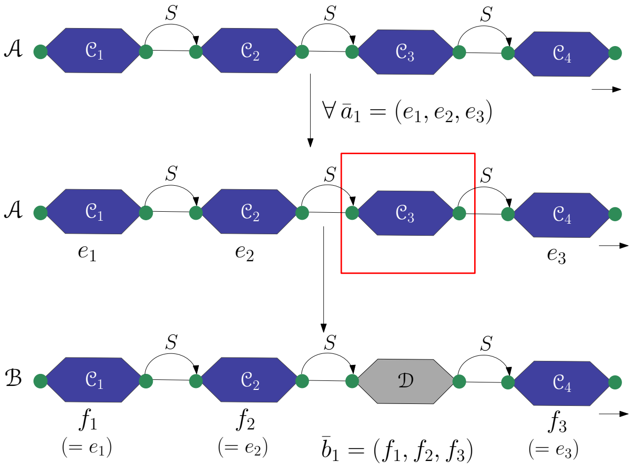
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{A}$ to $\mathbf{B}$



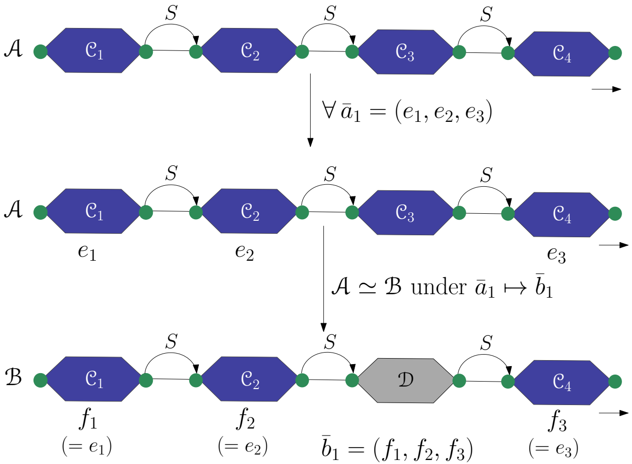
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{A}$ to $\mathbf{B}$



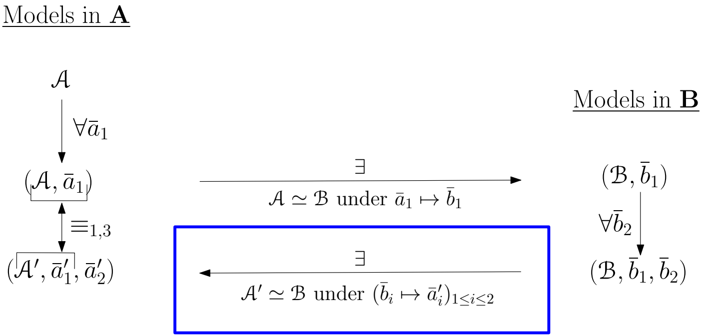
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{A}$ to $\mathbf{B}$



Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{A}$ to $\mathbf{B}$



Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{B}$ to $\mathbf{A}$



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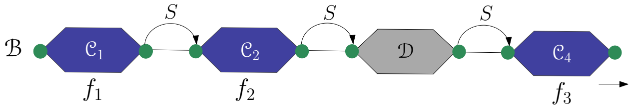
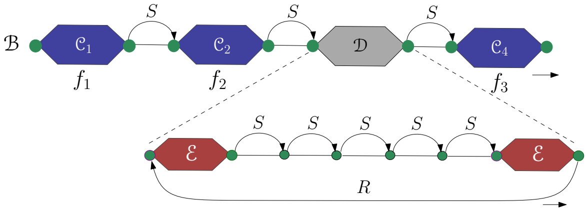
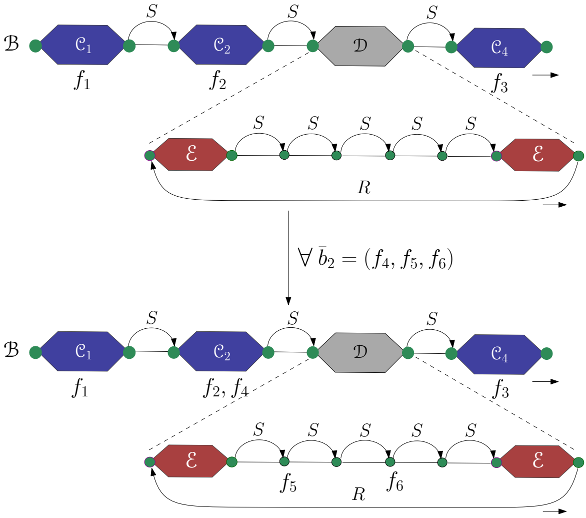
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{B}$ to $\mathbf{A}$



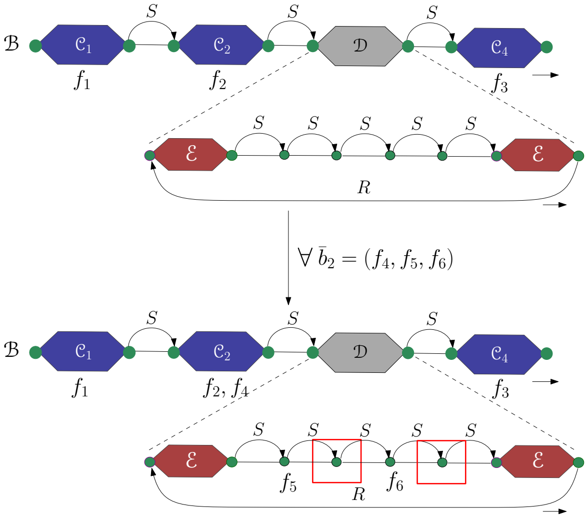
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{B}$ to $\mathbf{A}$



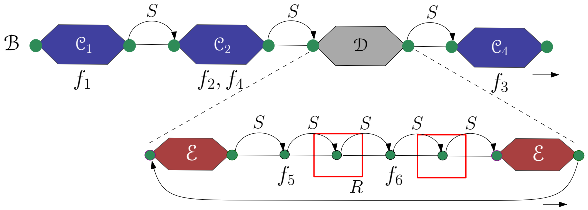
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{B}$ to $\mathbf{A}$



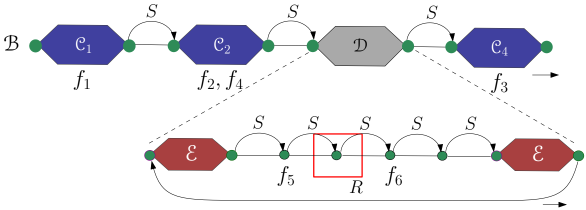
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{B}$ to $\mathbf{A}$



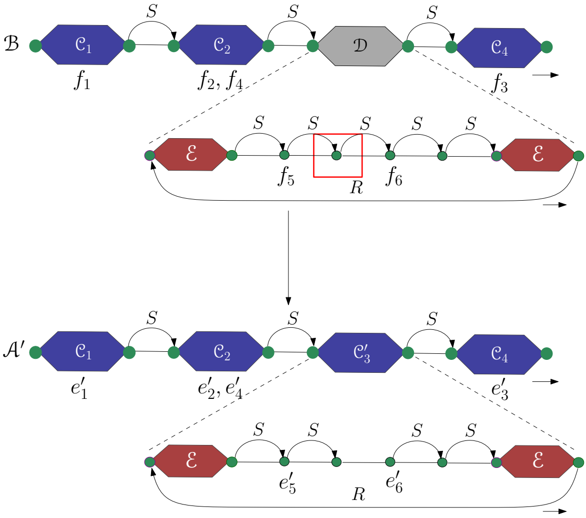
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{B}$ to $\mathbf{A}$



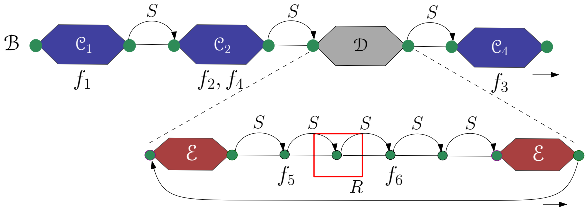
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{B}$ to $\mathbf{A}$



Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{B}$ to $\mathbf{A}$

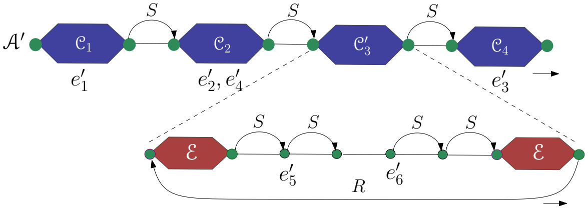


Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: From $\mathbf{B}$ to $\mathbf{A}$

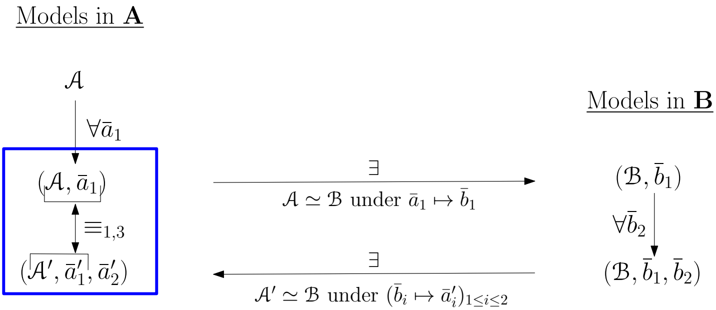


$\bar{a}'_1 = (e'_1, e'_2, e'_3) = \bar{b}_1$
$\bar{a}'_2 = (e'_4, e'_5, e'_6) = \bar{b}_2$

$B \simeq A'$ under $(\bar{b}_i \mapsto \bar{a}'_i)_{1 \leq i \leq 2}$

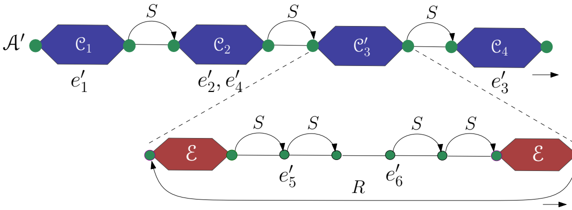


Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: Showing $(\mathcal{A}, \bar{a}_1) \equiv_{1,3} (\mathcal{A}', \bar{a}'_1)$

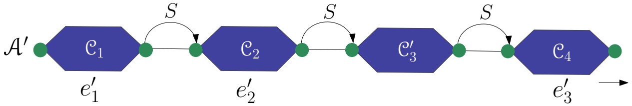


"$\mathcal{A} \simeq \mathcal{B}$ under $\bar{a} \mapsto \bar{b}$" $:=$ $\bar{a} \mapsto \bar{b}$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

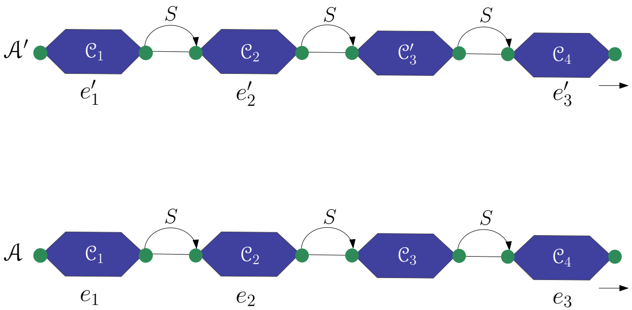
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: Showing $(\mathcal{A}, \bar{a}_1) \equiv_{1,3} (\mathcal{A}', \bar{a}'_1)$



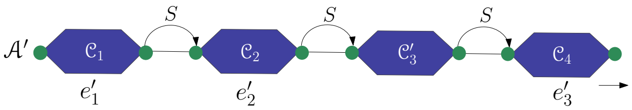
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: Showing $(\mathcal{A}, \bar{a}_1) \equiv_{1,3} (\mathcal{A}', \bar{a}_1')$



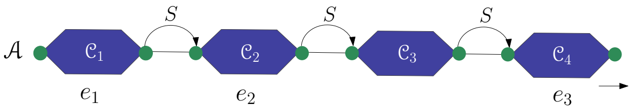
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: Showing $(\mathcal{A}, \bar{a}_1) \equiv_{1,3} (\mathcal{A}', \bar{a}'_1)$



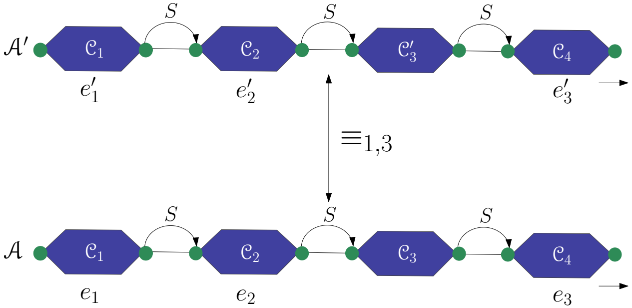
Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: Showing $(\mathcal{A}, \bar{a}_1) \equiv_{1,3} (\mathcal{A}', \bar{a}'_1)$



$$e'_1 = e_1, e'_2 = e_2, e'_3 = e_3$$



Executing proof approach for $\mathbf{A} \Rrightarrow_{2,3} \mathbf{B}$: Showing $(\mathcal{A}, \bar{a}_1) \equiv_{1,3} (\mathcal{A}', \bar{a}'_1)$



Inexpressibility in Σ_3

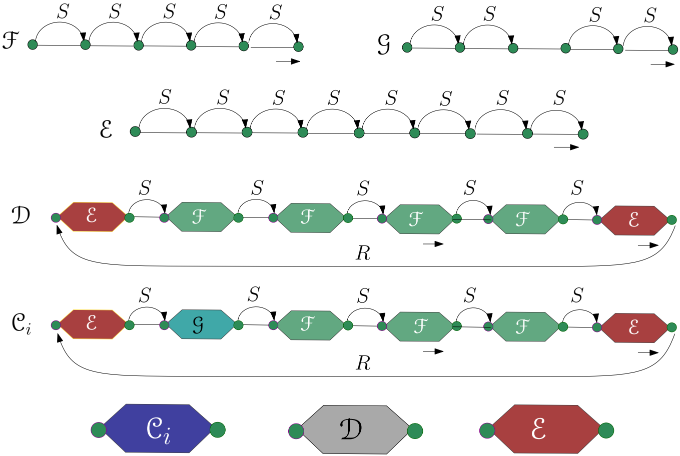
Even stronger failure of Łoś-Tarski theorem in the finite

Proposition

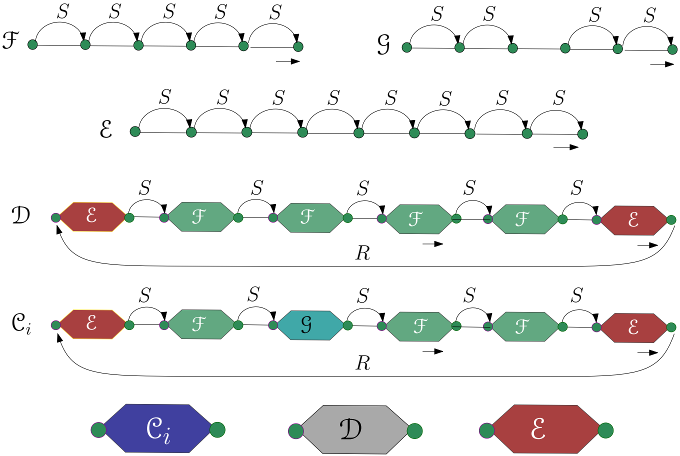
The sentence Ψ, which is hereditary over all finite structures, is not equivalent over this class to any Σ_3 sentence.

- Let $\Sigma_{3,k}$ = class of Σ_3 sentences in which each block of quantifiers has size k.
- For each k, we construct a class $\mathbf{A}$ of models and a class $\mathbf{B}$ of non-models of Ψ such that $\mathbf{A} \Rrightarrow_{3,k} \mathbf{B}$ holds: for each $\Sigma_{3,k}$ sentence θ, if $\mathbf{A}$ contains a model of θ, then so does $\mathbf{B}$.
- We illustrate our constructions for $k = 3$.

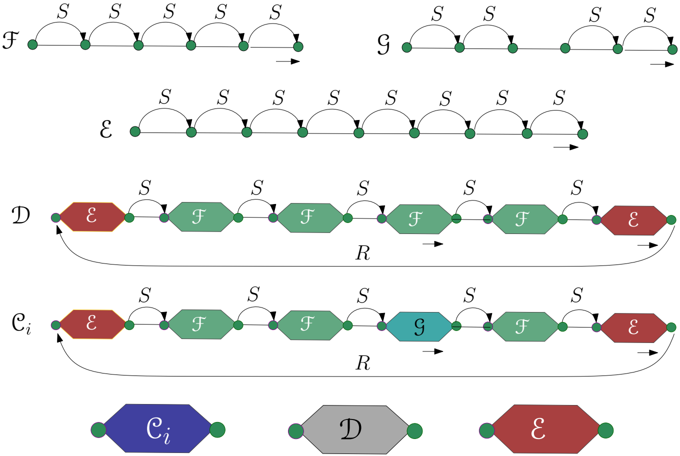
Construction of classes $\mathbf{A}$ and $\mathbf{B}$



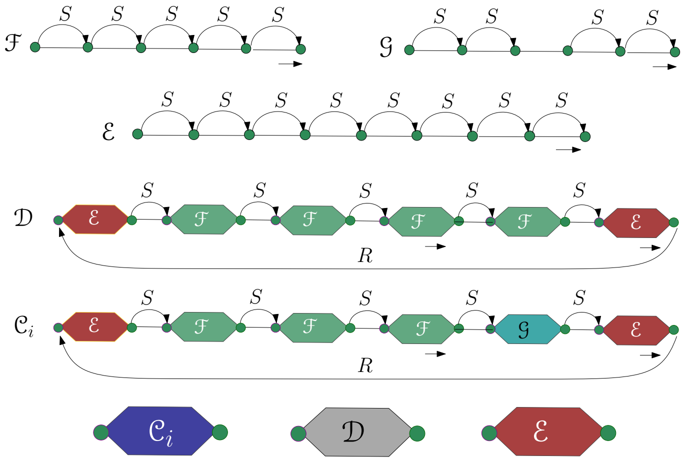
Construction of classes $\mathbf{A}$ and $\mathbf{B}$



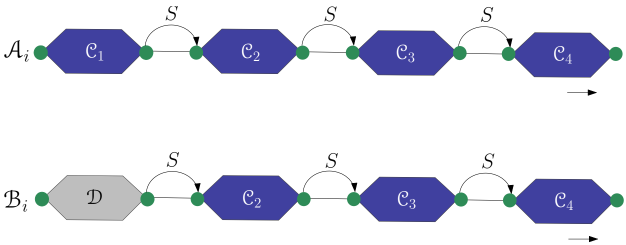
Construction of classes $\mathbf{A}$ and $\mathbf{B}$



Construction of classes $\mathbf{A}$ and $\mathbf{B}$

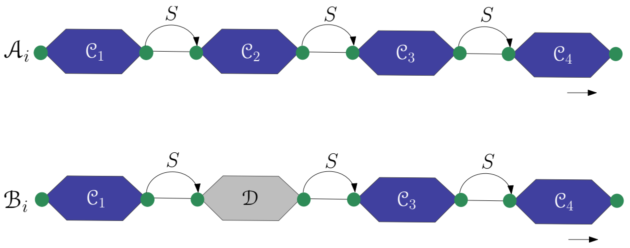


Construction of classes $\mathbf{A}$ and $\mathbf{B}$



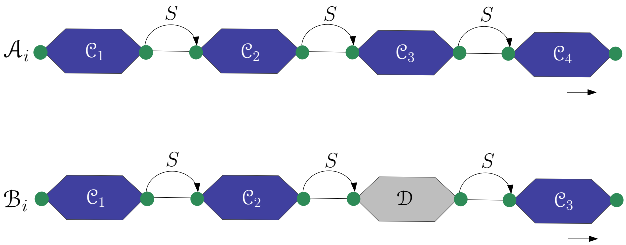
$\mathbf{A}$ = Class of all $\mathcal{A}_i$s; $\mathbf{B}$ = Class of all $\mathcal{B}_i$s

Construction of classes $\mathbf{A}$ and $\mathbf{B}$



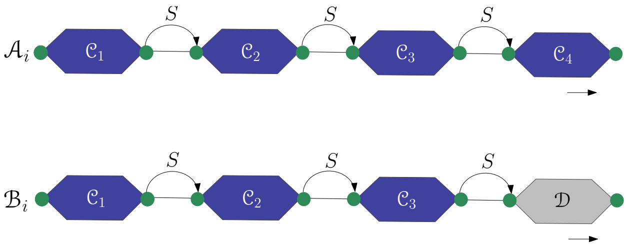
$\mathbf{A}$ = Class of all $\mathcal{A}_i$s; $\mathbf{B}$ = Class of all $\mathcal{B}_i$s

Construction of classes $\mathbf{A}$ and $\mathbf{B}$



$\mathbf{A}$ = Class of all $\mathcal{A}_i$s; $\mathbf{B}$ = Class of all $\mathcal{B}_i$s

Construction of classes $\mathbf{A}$ and $\mathbf{B}$



$\mathbf{A}$ = Class of all $\mathcal{A}_i$s; $\mathbf{B}$ = Class of all $\mathcal{B}_i$s

Similarly as before, it can be shown that $\mathcal{A}_i \models \Psi$ and $\mathcal{B}_i \models \neg\Psi$.

Towards showing $\mathbf{A} \Rrightarrow_{3,3} \mathbf{B}$

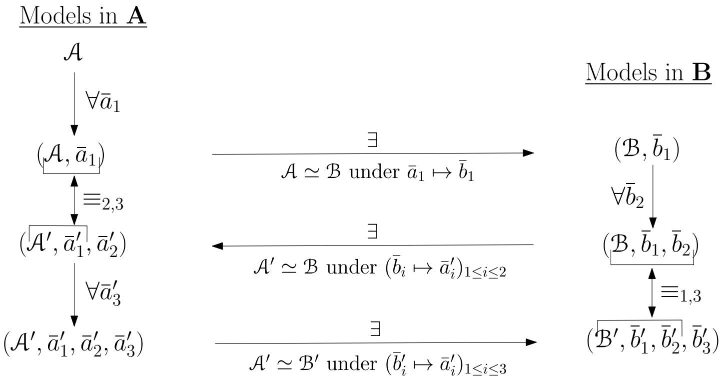
- For structures $\mathcal{M}_1$ and $\mathcal{M}_2$, denote by $\mathcal{M}_1 \equiv_{2,3} \mathcal{M}_2$ that $\mathcal{M}_1$ and $\mathcal{M}_2$ agree on all sentences of $\Sigma_{2,3}$.

Lemma

The following equivalences hold for any i, j:

- $\mathcal{C}_i \equiv_{2,3} \mathcal{C}_j$
- $\mathcal{A}_i \equiv_{2,3} \mathcal{A}_j$
- $(\mathcal{A}, \bar{a}) \equiv_{2,3} (\mathcal{A}', \bar{a})$ where $\bar{a}$ is a 3-tuple and $\mathcal{A}'$ is obtained from $\mathcal{A}$ by replacing the $\mathcal{C}_i$ segment not touched by $\bar{a}$, with $\mathcal{C}_j$

Overview of proof approach for showing $\mathbf{A} \Rrightarrow_{3,3} \mathbf{B}$



Expressibility in Datalog($\neq, \neg$)

Datalog syntax

- A Datalog($\neq, \neg$) rule is of one of the foll. forms:

$$\begin{array}{rcl} R(\bar{x}) & \longleftarrow & A(\bar{x}_1) \\ R(\bar{x}) & \longleftarrow & R_1(\bar{x}_1), \ldots, R_n(\bar{x}_n) \end{array}$$

- In the first rule above, $A(\bar{x}_1)$ is an atom that can appear negated. Also A can be equality or its negation.
- In the second rule above, all predicates R_i that are not atoms appear un-negated. Also, R can be one of the R_is.
- In both rules, the variables appearing in the LHS are a subset of the variables appearing in the RHS.
- A Datalog($\neq, \neg$) program is a finite set of Datalog rules.

Datalog model-theoretic semantics

- Consider the following Datalog program:

$$\begin{array}{rcl} R(x,y) & \longleftarrow & A(x,z), B(z,y) \\ R(x,y) & \longleftarrow & \neg A(x,z), R(x,y) \end{array}$$

- The first rule as a program by itself corresponds to

$$\alpha(x,y) := \exists z (A(x,z) \wedge B(z,y))$$

- With both rules, the program corresponds to the existential least fixpoint logic sentence $\beta(x,y)$ given as below:

$$\begin{array}{rcl} \beta(x,y) & := & \mathsf{LFP}_{R,u,v}\varphi(R,u,v)](x,y) \\ \varphi(R,u,v) & := & \alpha(u,v) \vee \exists z(\neg A(u,z) \wedge R(u,v)) \end{array}$$

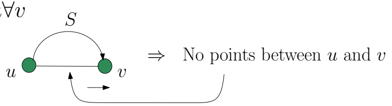
- Datalog($\neq, \neg$) corresponds exactly to existential least fixpoint logic, and thus any Datalog($\neq, \neg$) program is extension closed.

$\neg\Psi$ as a Datalog program

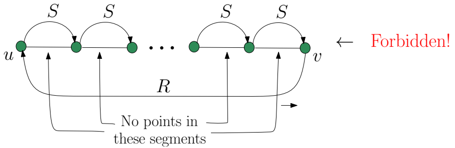
$$\Psi := \xi_1 \wedge \xi_2 \wedge \xi_3 \quad \in \mathrm{FO}(\sigma) \text{ where } \sigma = \{\leq, R, S\}$$

$\xi_1 :=$ " $\leq$ is a linear order"

$\xi_2 := \forall u \forall v$



$\xi_3 := \forall u \forall v$



- Express $\neg\xi_1, \neg\xi_2, \neg\xi_3$ as Datalog programs $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ with "start symbols" T_1, T_2, T_3 resp. Then the Datalog program for $\neg\Psi$ is

$$T \quad \longleftarrow \quad T_1 \mid T_2 \mid T_3$$

$\neg\xi_1$, $\neg\xi_2$ and $\neg\xi_3$ as Datalog programs

$\xi_1 :=$ " $\leq$ is a linear order"

$$\xi_1 := \forall x \forall y \forall z \qquad \begin{pmatrix} x \leq x \;\; \wedge \\ (x \leq y \;\; \wedge \;\; y \leq x) \rightarrow \;\; x = y \;\; \wedge \\ (x \leq y \;\; \wedge \;\; y \leq z) \rightarrow \;\; x \leq z \end{pmatrix}$$

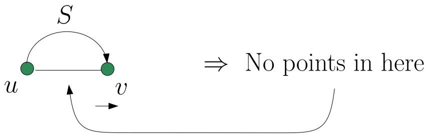
$$\neg\xi_1 := \exists x \exists y \exists z \qquad \begin{pmatrix} \neg\; x \leq x & \vee \\ (x \leq y \wedge y \leq x \wedge x \neq y) & \vee \\ (x \leq y \wedge y \leq z \wedge \neg\; x \leq z) & \end{pmatrix}$$

Datalog program for $\neg\xi_1$:

$$\begin{aligned} T_1 \longleftarrow \; & \neg x \leq x \mid \\ & x \leq y, \; y \leq x, \; x \neq y \;\mid \\ & x \leq y, \; y \leq z, \; \neg\; x \leq z \end{aligned}$$

$\neg\xi_1$, $\neg\xi_2$ and $\neg\xi_3$ as Datalog programs

$\xi_2 := \forall u \forall v$



$$\xi_2 := \forall u \forall v \; S(u,v) \rightarrow \neg \exists z \begin{pmatrix} u \leq z \wedge z \leq v \;\; \wedge \\ u \neq z \wedge z \neq v \end{pmatrix}$$

$$\neg\xi_2 := \exists u \exists v \; S(u,v) \wedge \exists z \begin{pmatrix} u \leq z \wedge z \leq v \;\; \wedge \\ u \neq z \wedge z \neq v \end{pmatrix}$$

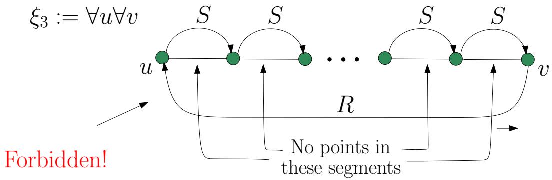
Datalog program for $\neg\xi_2$:

$$T_2 \longleftarrow S(u,v), X(u,v)$$

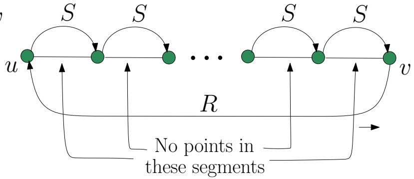
$$X(u,v) \longleftarrow u \leq z, z \leq v, u \neq z, z \neq v$$

$\neg\xi_1$, $\neg\xi_2$ and $\neg\xi_3$ as Datalog programs

$\xi_3 := \forall u \forall v$



$\neg\xi_3 := \exists u \exists v$



Datalog program for $\neg\xi_3$:

$$T_3 \longleftarrow R(v,u), S\text{-reach}(u,v)$$

$$S\text{-reach}(u,v) \longleftarrow S(u,v) \;\mid\; S(u,z), S\text{-reach}(z,v)$$

Generalizing Tait's sentence

Overview

Theorem

For every n, there is a vocabulary τ_n and an FO(τ_n) sentence φ_n such that the following hold:

1. φ_n is hereditary over all finite structures, but is not equivalent over this class to any Σ_n sentence.
2. $\neg\varphi_n$ can be expressed in Datalog$(\neq, \neg)$.

- Construction of φ_n
- Inexpressibility of φ_n in Σ_n using a suitable class of models and non-models

Construction of φ_n

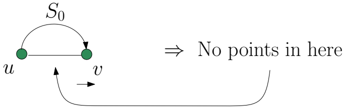
Construction of φ_n

Consider $\Psi_1(x, y)$ over $\sigma_1 = \{\leq, R_1, S_0, S_1\}$ as below.

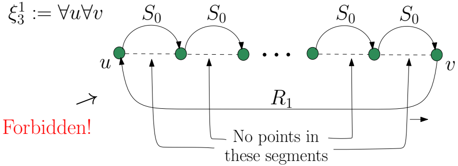
$$\Psi_1(x, y) := S_1(x, y) \to \left(\xi_1^1(x, y) \wedge \xi_2^1 \wedge \xi_3^1\right)$$

$\xi_1^1(x, y) :=$ " $\leq$ is a linear order" $\wedge$
"x is min and y is max under $\leq$"

$\xi_2^1 := \forall u \forall v$



$\xi_3^1 := \forall u \forall v$



Construction of φ_n

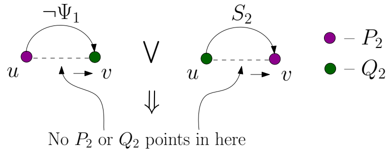
Consider $\Psi_2(x, y)$ over $\sigma_2 = \sigma_1 \cup \{P_2, Q_2, R_2, S_2\}$ as below.

$$\Psi_2(x, y) := S_2(x, y) \rightarrow \left(\xi_1^2(x, y) \wedge \xi_2^2 \wedge \xi_3^2\right)$$

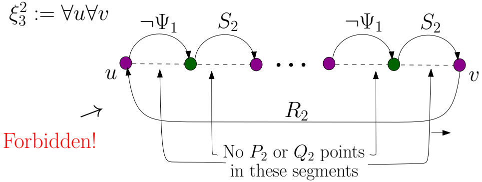
$\xi_1^2(x, y) :=$ " $\leq$ is a linear order" $\wedge$

"x is min and y is max under $\leq$ " $\wedge\ P_2(x) \wedge Q_2(y)$

$\xi_2^2 := \forall u \forall v$



$\xi_3^2 := \forall u \forall v$



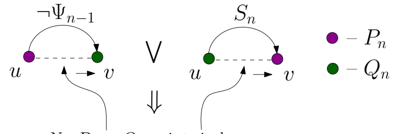
Construction of φ_n

Consider $\Psi_n(x, y)$ over $\sigma_n = \sigma_{n-1} \cup \{P_n, Q_n, R_n, S_n\}$ as below.

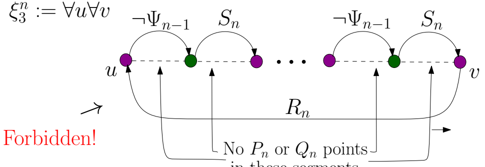
$$\Psi_n(x,y) := S_n(x,y) \rightarrow \left(\xi_1^n(x,y) \wedge \xi_2^n \wedge \xi_3^n\right)$$

$\xi_1^n(x,y) :=$ " $\leq$ is a linear order" $\wedge$
"x is min and y is max under $\leq$" $\wedge P_n(x) \wedge Q_n(y)$

$\xi_2^n := \forall u \forall v$



$\xi_3^n := \forall u \forall v$



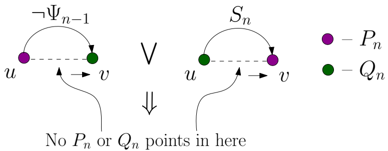
Construction of φ_n

$$\varphi_n := \Psi_n(x,y)[x \mapsto c; y \mapsto d] \text{ for constants } c, d$$

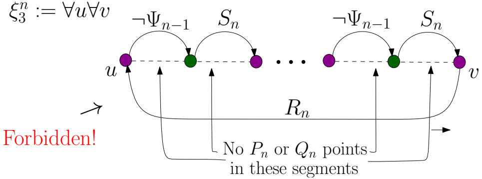
$$\Psi_n(x,y) := S_n(x,y) \to \left(\xi_1^n(x,y) \wedge \xi_2^n \wedge \xi_3^n\right)$$

$$\xi_1^n(x,y) := \text{“} \leq \text{ is a linear order”} \wedge \text{“}x \text{ is min and } y \text{ is max under } \leq \text{”} \wedge P_n(x) \wedge Q_n(y)$$

$\xi_2^n := \forall u \forall v$



$\xi_3^n := \forall u \forall v$



Inexpressibility of φ_n in Σ_{2n}

Theorem

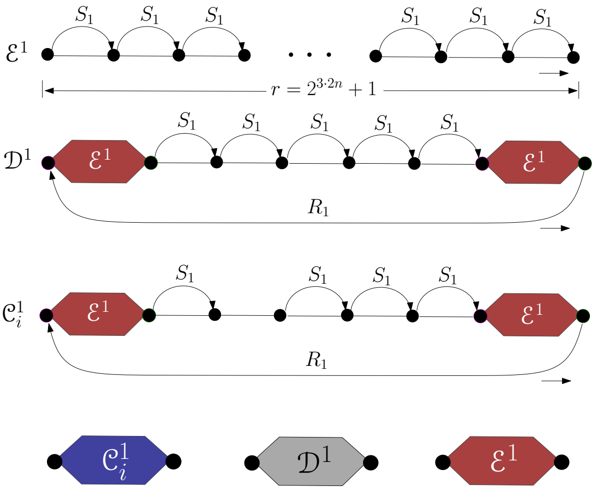
The following are true of $\Psi_n(x, y)$, and therefore also of φ_n:

1. $\neg\Psi_n(x, y)$ is expressible in Datalog$(\neq, \neg)$ and hence $\Psi_n(x, y)$ is hereditary over all finite structures.
2. $\Psi_n(x, y)$ is not equivalent to any Σ_{2n} formula.

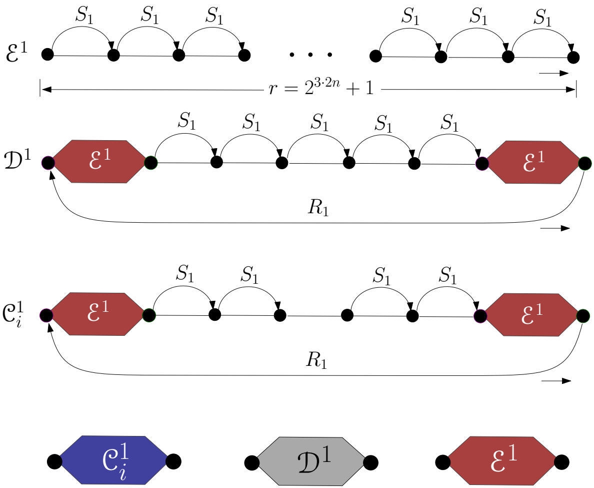
- Define $\Sigma_{2n,k}$ analogously to $\Sigma_{2,k}$ and for formulae.
- For each k, we construct a class $\mathbf{A}^n$ of models and a class $\mathbf{B}^n$ of non-models of $\Psi_n(x, y)$ such that $\mathbf{A}^n \Rrightarrow_{2n,k} \mathbf{B}^n$ holds: for each $\Sigma_{2n,k}$ formula $\theta(x, y)$, if $\mathbf{A}^n$ contains a model of $\theta(x, y)$, then so does $\mathbf{B}^n$.
- We again illustrate our constructions for $k = 3$.

Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$

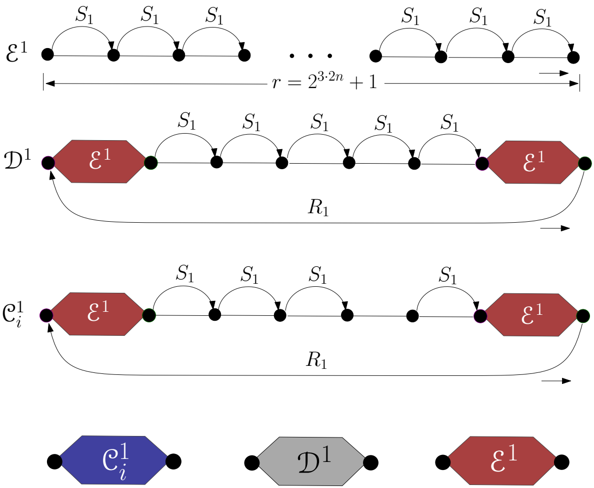
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level 1



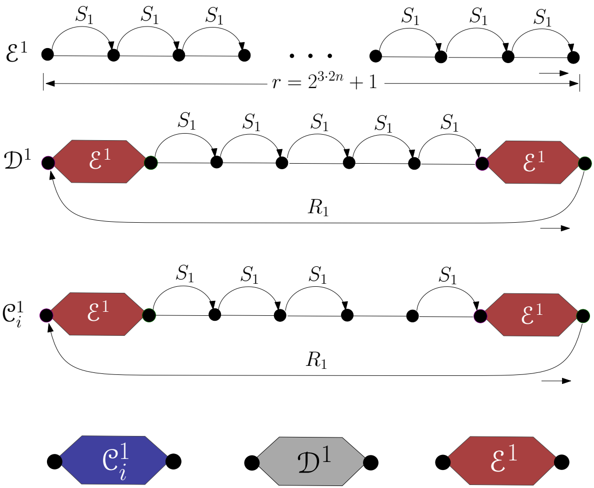
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level 1



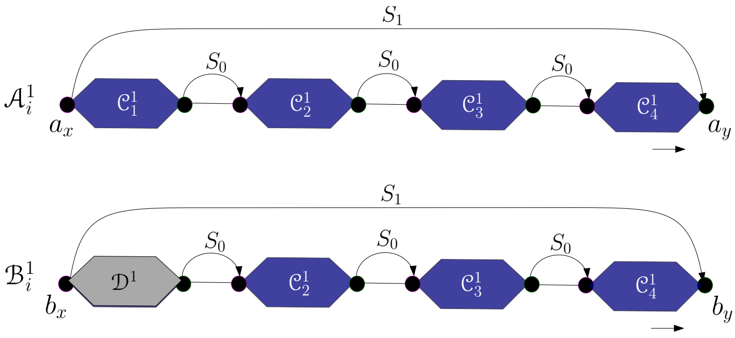
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level 1



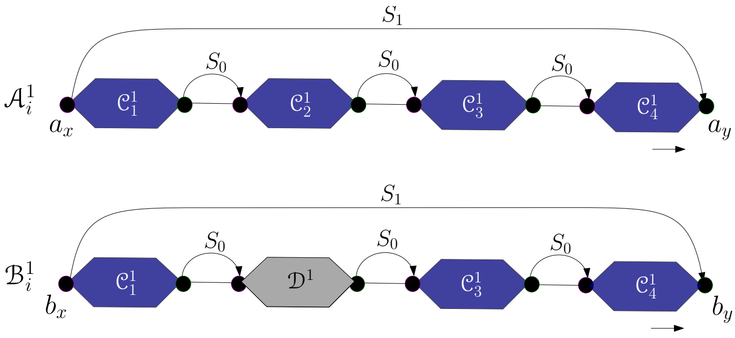
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level 1



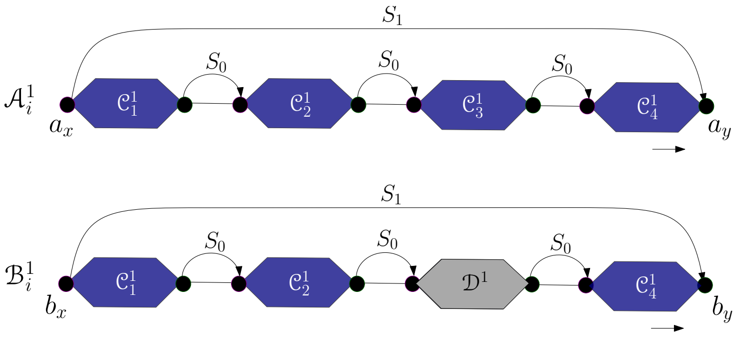
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level 1



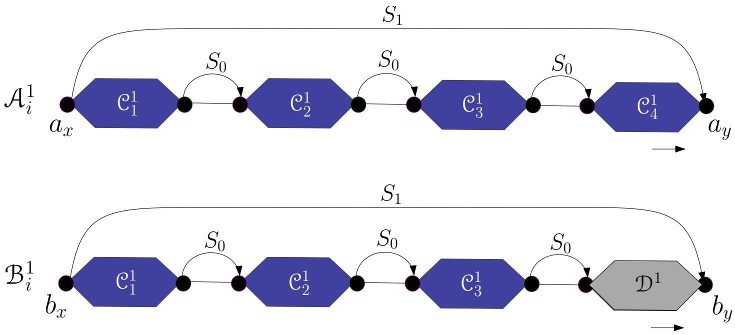
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level 1



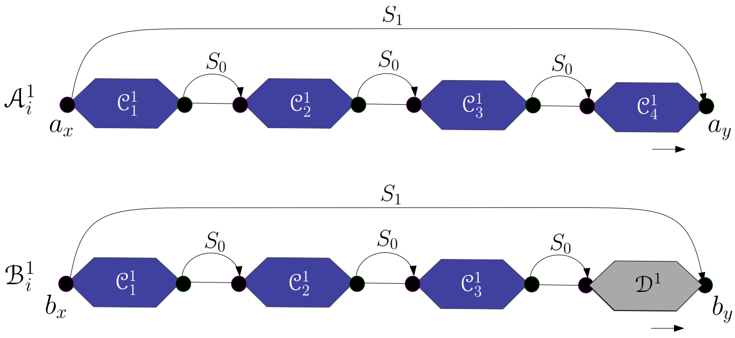
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level 1



Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level 1

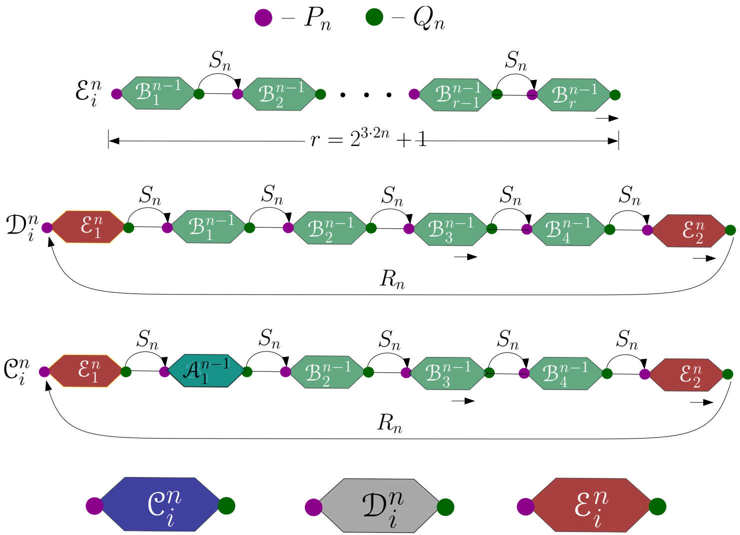


Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level 1

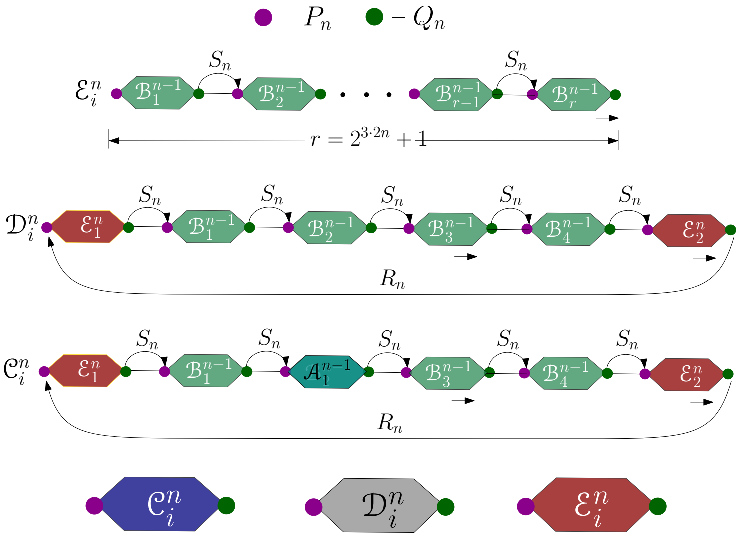


$\mathbf{A}^1$ = Class of all $\mathcal{A}_i^1$s; $\mathbf{B}^1$ = Class of all $\mathcal{B}_i^1$s

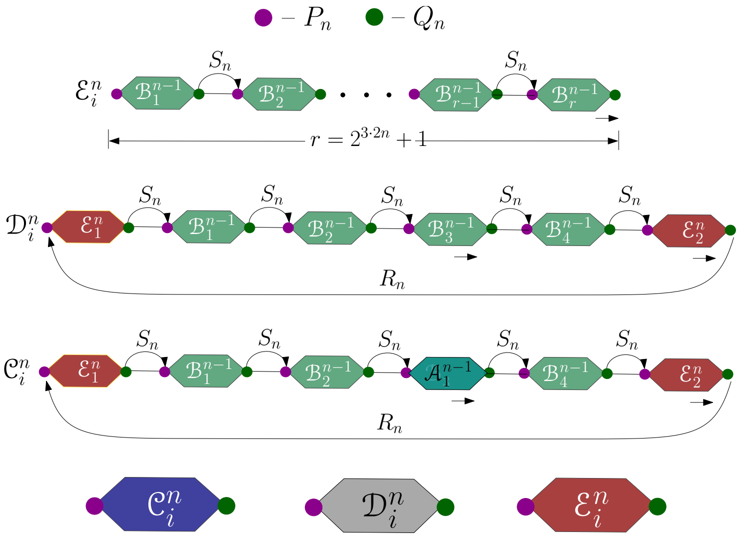
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level n



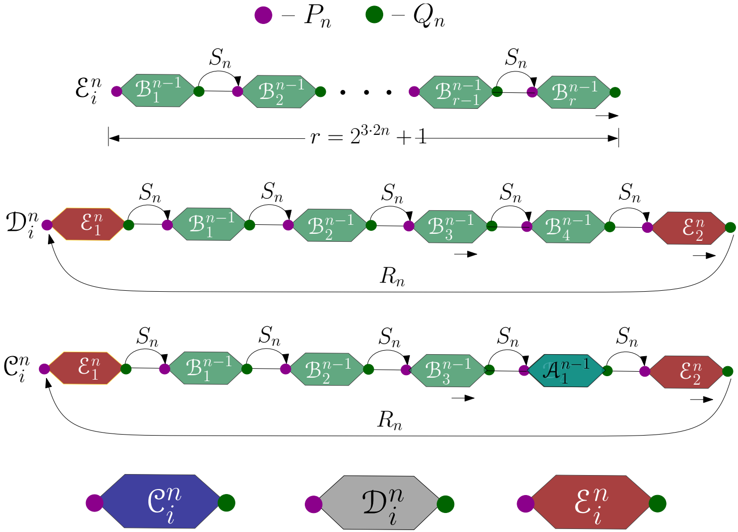
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level n



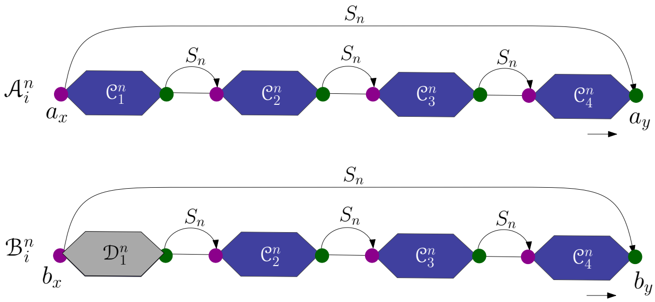
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level n



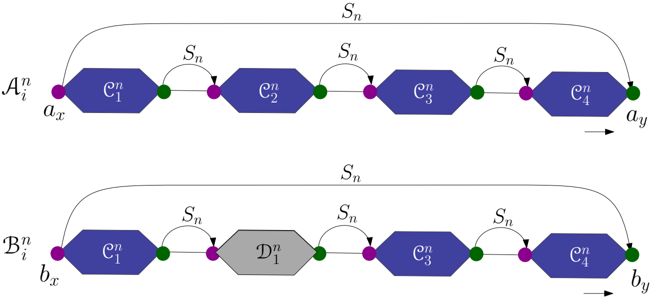
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level n



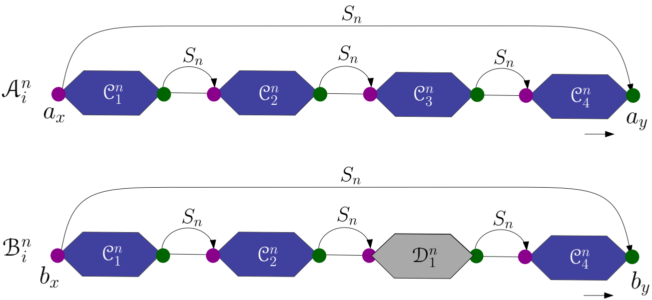
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level n



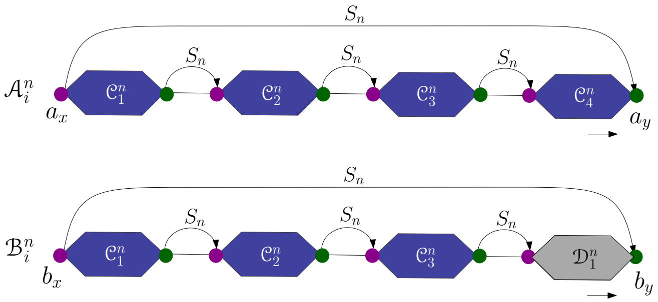
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level n



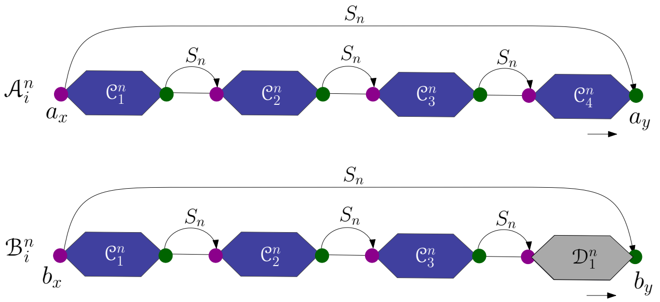
Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level n



Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level n



Construction of $\mathbf{A}^n$ and $\mathbf{B}^n$: Structures at level n



$\mathbf{A}^n$ = Class of all $\mathcal{A}_i^n$s; $\mathbf{B}^n$ = Class of all $\mathcal{B}_i^n$s

It can be shown that

$$(\mathcal{A}_i^n, a_x, a_y) \models \Psi_n(x, y)$$
$$(\mathcal{B}_i^n, b_x, b_y) \models \neg\Psi_n(x, y)$$

Conclusion

Future directions

- The formula Ψ_n is over a vocabulary σ_n that grows with n.
- Further, σ_n can be seen as the vocabulary of ordered vertex colored graphs with multiple edge relations.

Open question 1.

Is there a fixed (finite) vocabulary σ^* such that prefix classes fail to capture FO(σ^*) expressible hereditary properties?

Open question 2.

Do prefix classes fail to capture hereditary properties of undirected graphs (possibly vertex colored)? In particular, does Łoś-Tarski theorem continue to fail over these graphs?

Thank you!