A Generalization of the Łoś-Tarski Preservation Theorem via Characterizations of Σ_n and Π_n Theories

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Introduction

- A preservation theorem characterizes (definable) classes of structures closed under a given model theoretic operation.
- Preservation under substructures/extensions – the Łoś-Tarski theorem (1949-50).
- The Łoś-Tarski theorem gives semantic characterizations of Π_1 and Σ_1 theories.
- Subsequent characterizations of Π_2 theories in terms of preservation under of ascending chains, intersections of descending chains, intersections of submodels, etc.
- Uniform set of preservation theorems characterizing Π_n and Σ_n theories for all $n \geq 1$, provided by H. J. Keisler in 1960.

Talk Outline and Assumptions for the talk

Talk Outline:

- The Łoś-Tarski theorem
- Σ_n-substructures/Σ_n-extensions
- The generalized Łoś-Tarski theorem
- Keisler's theorem
- Comparison

Assumptions:

- First Order (FO) theories.
- Arbitrary vocabularies (constants, predicates and functions)
- Arbitrary structures

Basic notions from model theory

- $\Sigma_1 = \exists^*(\ldots), \Pi_1 = \forall^*(\ldots)$
  $\Sigma_n = \exists^*\forall^* \cdots (n\text{-blocks}); \Pi_n = \forall^*\exists^* \cdots (n\text{-blocks})$
- Substructure/Extension, $\mathfrak{A} \subseteq \mathfrak{B}$: similar to induced subgraph.
- Isomorphism, $\mathfrak{A} \cong \mathfrak{B}$: Standard notion.
- Elementary equivalence, $\mathfrak{A} \equiv \mathfrak{B}$: Indistinguishable by FO.
- Elementary substructure/elementary extension, $\mathfrak{A} \preceq \mathfrak{B}$: (i) $\mathfrak{A} \subseteq \mathfrak{B}$ and (ii) for every FO formula $\varphi(x_1, \ldots, x_n)$ and every n-tuple $\bar{a}$ from $\mathfrak{A}$, we have $\mathfrak{A} \models \varphi(\bar{a})$ iff $\mathfrak{B} \models \varphi(\bar{a})$.
- $\mathfrak{A} \Rrightarrow_n \mathfrak{B}$: every Σ_n sentence true in $\mathfrak{A}$ is also true in $\mathfrak{B}$.
- Σ_n-theory($\mathfrak{A}$): set of all Σ_n sentences true in $\mathfrak{A}$.

The Łoś-Tarski theorem

Preservation under Substructures

Definition 1 (Pres. under subst.)

A theory T is said to be **preserved under substructures**, *abbreviated T* **is** *PS, if $\big((M \models T) \wedge (N \subseteq M)\big) \rightarrow N \models T$.*

- E.g.: Consider $\phi_n = \neg\exists x_1 \ldots \exists x_n(E(x_1, x_2) \wedge \ldots E(x_n, x_1))$ expresses the absence of a cycle of length n.
- Then $T = \{\phi_n \mid n \geq 1\}$ expresses acyclicity. Any induced subgraph of an acyclic graph is also acyclic. Then T is PS.
- In general, every Π_1 theory is PS.

Theorem 1 (Łoś-Tarski, 1949-50)

A FO theory is PS iff it is equivalent to a Π_1 theory.

Preservation under Extensions

Definition 2 (Pres. under ext.)

A theory T *is said to be* **preserved under extensions**, *abbreviated* T **is** PE, *if* $\big((M \models T) \wedge (M \subseteq N)\big) \rightarrow N \models T$.

- E.g.: If $\psi_n = \neg\phi_n$ where ϕ_n is as before, then $T = \{\psi_n \mid n \geq 1\}$ asserts the presence of cycles of all lengths. Clearly T is PE.
- In general, every Σ_1 theory is PE.

Theorem 2 (Łoś-Tarski, 1949-50)

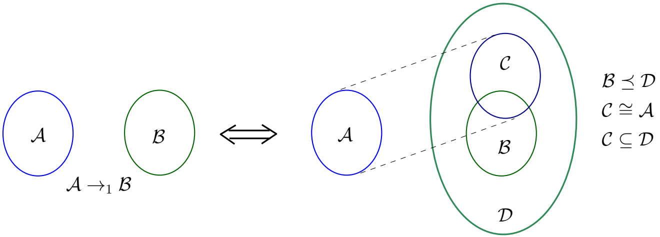
A FO theory is PE *iff it is equivalent to a* Σ_1 *theory.*

Proof Sketch of Łoś-Tarski Theorem

Following is a key theorem used in the proof.

Theorem 3 (Existential Amalgamation Theorem)

$\mathfrak{A} \Rrightarrow_1 \mathfrak{B}$ *iff there exist structures* $\mathfrak{C}$ *and* $\mathfrak{D}$ *s.t. (i)* $\mathfrak{B} \preceq \mathfrak{D}$ *(ii)* $\mathfrak{C} \cong \mathfrak{A}$ *and (iii)* $\mathfrak{C} \subseteq \mathfrak{D}$.



Proof Sketch of Łoś-Tarski Theorem

(Proof of subst. version)

- Suppose T is PS. Let Γ = set of all Π_1 consequences of T. We show that $T \leftrightarrow \Gamma$. Clearly T entails Γ.
- Conversely, suppose $\mathfrak{A} \models \Gamma$. Show that there is a model $\mathfrak{B}$ of T s.t. $\mathfrak{A} \Rrightarrow_1 \mathfrak{B}$. (by showing that $T \cup \Sigma_1$-theory$(\mathfrak{A})$ is sat.)

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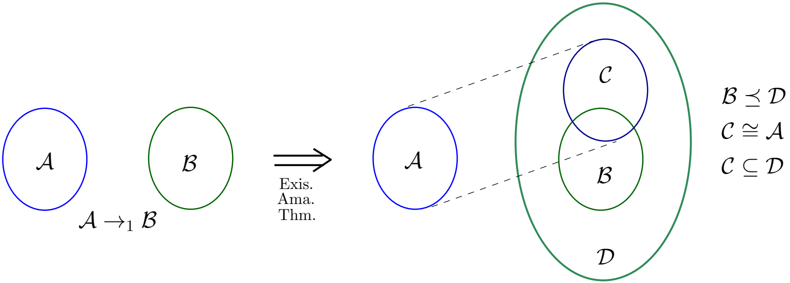

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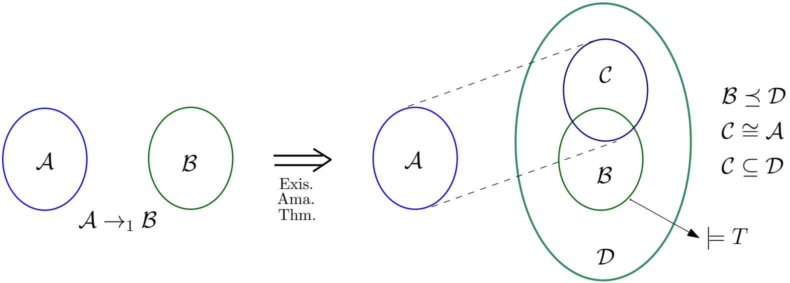

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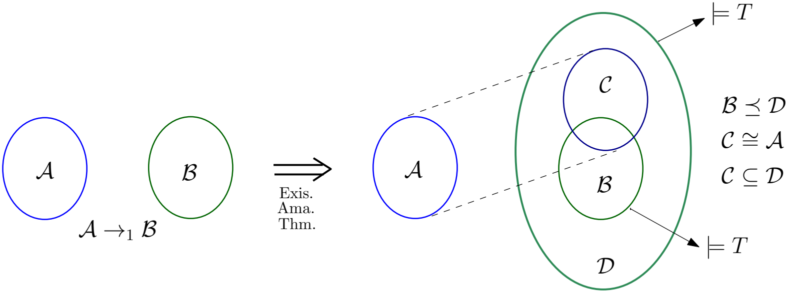

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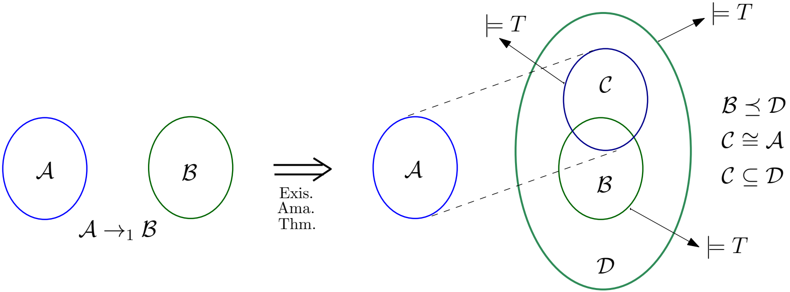

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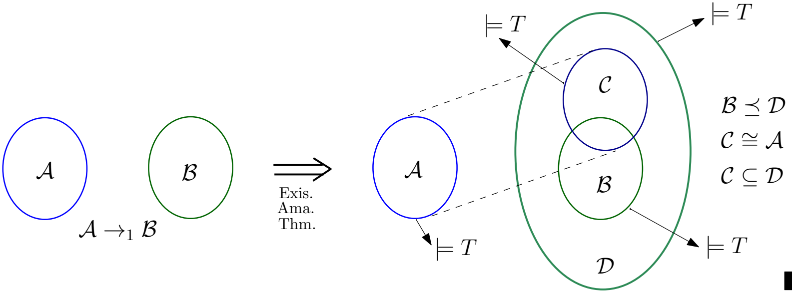

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Σ_n-substructures and Σ_n-extensions

Σ_n-substructures and Σ_n-extensions

- Suppose $\mathfrak{A} \subseteq \mathfrak{B}$. Then observe the following property of $\subseteq$: for every Σ_1 formula $\varphi(x_1, \ldots, x_r)$ and every r-tuple $\bar{a}$ from $\mathfrak{A}$, we have $\mathfrak{A} \models \varphi(\bar{a}) \rightarrow \mathfrak{B} \models \varphi(\bar{a})$.
- A natural generalization of the above property – consider Σ_n formulas instead of Σ_1 formulas.

Definition 3 (From literature)

We say $\mathfrak{A}$ is a Σ_n-substructure of $\mathfrak{B}$, denoted $\mathfrak{A} \subseteq_n \mathfrak{B}$ if (i) $\mathfrak{A} \subseteq \mathfrak{B}$ and (ii) for every Σ_n formula $\varphi(x_1, \ldots, x_r)$ and every r-tuple $\bar{a}$ from $\mathfrak{A}$, we have $\mathfrak{A} \models \varphi(\bar{a}) \rightarrow \mathfrak{B} \models \varphi(\bar{a})$. If $\mathfrak{A} \subseteq_n \mathfrak{B}$, then we say $\mathfrak{B}$ is a Σ_n-extension of $\mathfrak{A}$.

The Generalized Łoś-Tarski Theorem

Preservation under Σ_n-substructures and Preservation under Σ_n-extensions

Definition 4

A theory T is said to be

- **preserved under Σ_n-substructures**, *abbreviated T* **is** *PS_n, if $\big((M \models T) \wedge (N \subseteq_n M)\big) \rightarrow N \models T$.*
- **preserved under Σ_n-extensions**, *abbreviated T* **is** *PE_n, if $\big((M \models T) \wedge (M \subseteq_n N)\big) \rightarrow N \models T$.*

- Easy to check that (i) T is PS_1 iff T is PS (ii) T is PE_1 iff T is PE.
- Easy to check that (i) Π_n theories are PS_n (ii) Σ_n theories are PE_n.

Generalization of the Łoś-Tarski theorem

Theorem 4 (Substructural version)

A FO theory is PS_n iff it is equivalent to a Π_n theory.

Theorem 5 (Extensional version)

A FO theory is PE_n iff it is equivalent to a Σ_n theory.

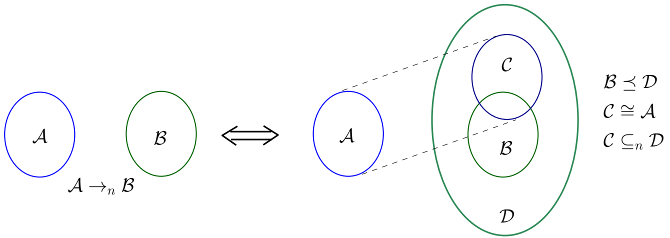
The case of $n = 1$ is exactly the Łoś-Tarski theorem.

Proof Sketch of the Generalized Łoś-Tarski Theorem

Following generalization of the Exis. Ama. Thm. is key for the proof.

Theorem 6 (Σ_n-Amalgamation Theorem)

$\mathfrak{A} \Rrightarrow_n \mathfrak{B}$ *iff there exist structures* $\mathfrak{C}$ *and* $\mathfrak{D}$ *s.t. (i)* $\mathfrak{B} \preceq \mathfrak{D}$ *(ii)* $\mathfrak{C} \cong \mathfrak{A}$ *and (iii)* $\mathfrak{C} \subseteq_n \mathfrak{D}$.



Proof Sketch of the Generalized Łoś-Tarski Theorem

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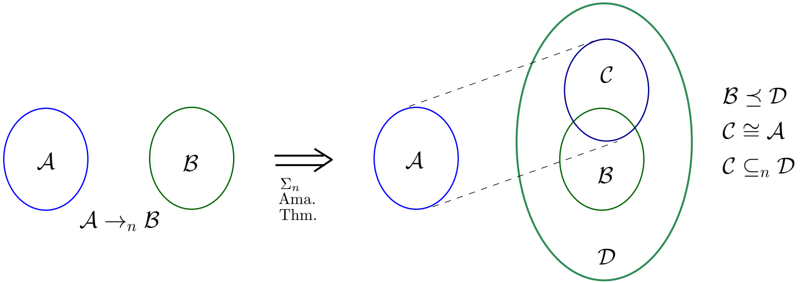

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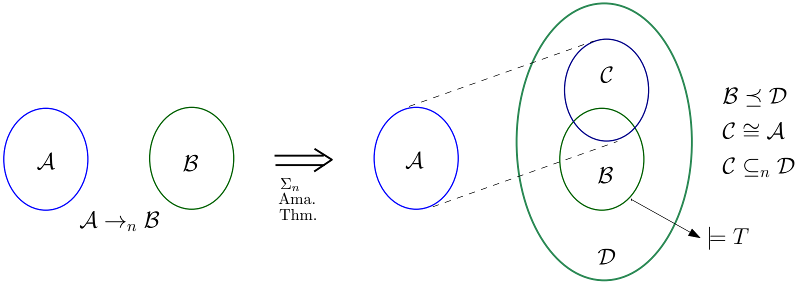

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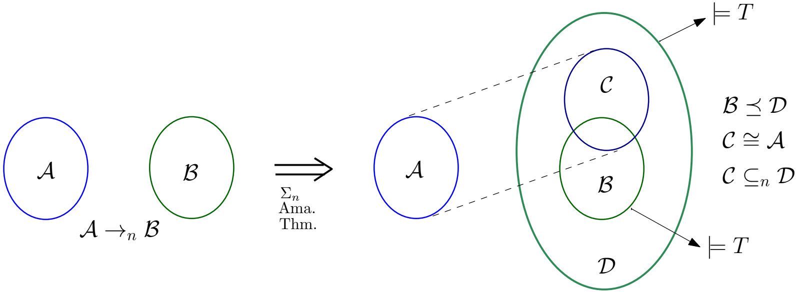

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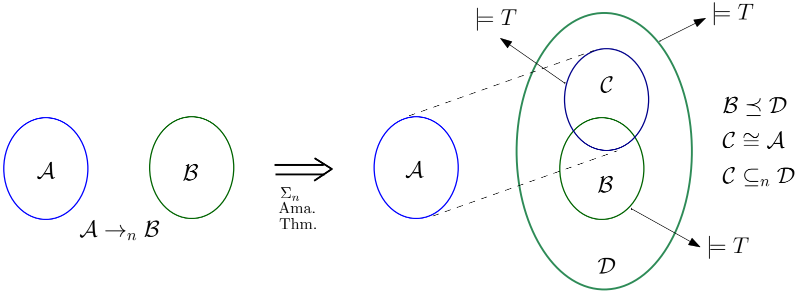

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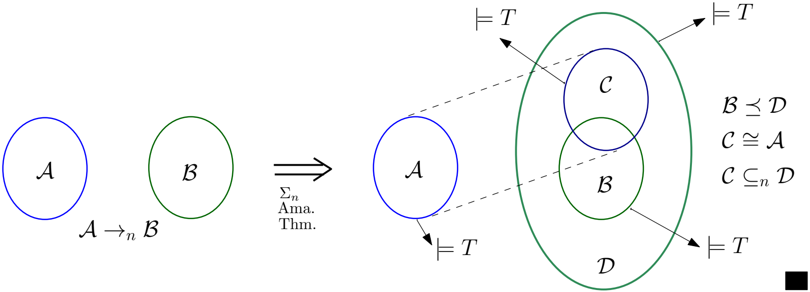

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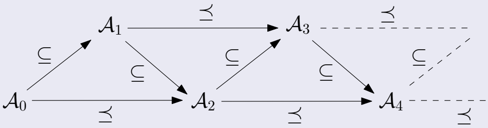
Keisler's Theorem

Keisler's notion of *sandwiches of finite orders*

Definition 5

For $0 < n < \omega$, a sequence of structures $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is said to be a **sandwich of order** n *if the structures are related as below:*



Definition 6

- $\mathfrak{B} \in n$-**Sand**$(\mathfrak{A})$ *iff* $\mathfrak{B}$ *and* $\mathfrak{A}$ *are (in order) the first two elements of a sandwich of order* n.
- $\mathfrak{B} \in n$-**Sand-by**$(\mathfrak{A})$ *iff* $\mathfrak{A}$ *and* $\mathfrak{B}'$ *are (in order) the first two elements of a sandwich of order* n, *for some* $\mathfrak{B}'$ *s.t.* $\mathfrak{B} \preceq \mathfrak{B}'$.

Preservation under n-Sand and Preservation under n-Sand-by

Definition 7

A theory T is **preserved under n-Sand (resp. n-Sand-by)** *if for all models $\mathfrak{A}$ of T, if $\mathfrak{B}$ belongs to n-Sand$(\mathfrak{A})$ (resp. n-Sand-by$(\mathfrak{A})$), then $\mathfrak{B}$ models T.*

- All Π_n sentences are pres. under n-Sand and Σ_n sentences are pres. under n-Sand-by.
- Observe that (i) T is pres. under 1-Sand iff T is PS (ii) T is pres. under 1-Sand-by iff T is PE.

Keisler's Generalization of Łoś-Tarski Theorem

Theorem 7

An FO theory is preserved under n-Sand iff it is equivalent to a Π_n theory.

Theorem 8

An FO theory is preserved under n-Sand-by iff it is equivalent to a Σ_n theory.

The case of $n = 1$ is exactly the Łoś-Tarski theorem.

Comparisons

Comparison between the two generalizations of the Łoś-Tarski theorem

- Clearly the notions of PS_n and PE_n are much simpler and easier to state than pres. under n-Sand and pres. under n-Sand-by.
- The proof of the generalization of the Łoś-Tarski theorem via PS_n and PE_n uses straightforward liftings of the ideas already contained in the standard proof of the Łoś-Tarski theorem.
- The proof of generalization via sandwiches is quite involved.

Relation between Σ_n-substructures and sandwiches of order n

Proposition 1

$\mathfrak{A} \subseteq_n \mathfrak{B}$ *iff* $\mathfrak{A} \in n$-*Sand*$(\mathfrak{B})$.

Following is a slight generalization of a result from [Chang-Keisler].

Proposition 2

$\mathfrak{A} \in n$-*Sand*$(\mathfrak{B})$ *iff there is a sequence of* $n+1$ *structures*

$$\mathfrak{A}_0 \subseteq \mathfrak{B}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{B}_1 \ldots$$

such that (i) $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{B}_0 = \mathfrak{B}$, *and (ii) each* $\mathfrak{B}_{k+1}$ *(resp.* $\mathfrak{A}_{k+1}$*) is isomorphic to an ultrapower of* $\mathfrak{B}_k$ *(resp.* $\mathfrak{A}_k$*), for* $0 \leq k \leq n$.

- Using the above propositions, we get a purely algebraic (i.e. without any reference to logic) formulation of $\subseteq_n$.

Thank You!

References I

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- W. Hodges, *A Shorter Model Theory*, Cambridge University Press, 1997.
- H. J. Keisler, Theory of models with generalized atomic formulas, *Journal of Symbolic Logic*, 25:1-26, 1960.